mer of the University of California gave an address by invitation on "The continued fraction representing cubic and higher irrationalities," and on Thursday morning Professor Tibor Radó of Ohio State University gave an address by invitation on "Recent work in the problem of Plateau." Sessions for the reading of papers were held on Tuesday and Wednesday afternoons and Friday morning.

The Mathematical Association held sessions Monday afternoon and Tuesday morning, President E. T. Bell presiding at both sessions. The Association is under obligations to the committee consisting of Professors W. M. Whyburn (chairman), E. T. Bell and R. M. Winger for the preparation of the instructive program. Abstracts of some of the papers follow, numbered in accordance with the numbers of the papers. Due to a cloudburst in the mountains, Professor C. A. Hutchinson failed to arrive in time for the meetings and his paper was read by title.

First Session of the Association

1. "Solid college geometry" by Professor N. A. Court, University of Oklahoma.

2. "The master's thesis" by Professor Mayme I. Logsdon, University of Chicago.

3. "Graeffe's method" by Professor C. A. Hutchinson, University of Colorado.

4. "Collegiate mathematics needed in the social sciences." A report of a committee of the Social Science Research Council, read by H. R. Tolley, Director of the Giannini Foundation of Agricultural Economics, University of California.

1. Professor Court made a plea for a course in synthetic solid geometry such as he is now giving to first year graduate students at the University of Oklahoma. He enumerated a number of relatively simple, but not commonly known, properties of the tetrahedron, which are readily proved by synthetic methods, these proofs having decided advantages over corresponding analytical proofs.

3. Professor Hutchinson pointed out that recent activity in various fields of applied science has renewed interest in the practical numerical solution of algebraic equations. The method due to Dandelin and Graeffe, published a century ago, is well adapted to the purpose. It has these advantages: (1) all roots, real and complex, are determined simultaneously; (2) the character of the roots appears early in the process used for their evaluation; (3) a calculating machine or multiplication tables can be used conveniently.

The process consists in deriving from the given equation a new one whose roots are high powers of the roots of the given equation. The roots of the transformed equation, being widely separated in absolute value, are then easily determined. Details of special cases are given in the paper.

4. This report will appear in full in an early number of this Monthly. The report called for a course including the following topics: logarithms, graphs (as a tool in the study of tabulated data), interpolation, equations and forms of

curves, probability, elements of differential and integral calculus, and curve fitting, these topics to be presented with particular reference to their application to the social sciences; these topics to be covered in a three-hour course running through two, or, at most, three semesters.

The discussion of this paper was quite enthusiastic and developed two different viewpoints, as indicated below in the abstract prepared by Professor James.

It was the sense of those speaking in behalf of the committee that such a selection of topics would decidedly shorten the time required to cover our ordinary courses extending through the elementary calculus. It was argued that experience in teaching brief courses in analysis justified this position. This viewpoint seemed to presuppose an intake ability on the part of the students, which if not constant is certainly not dependent upon the conventional sequences of topics.

On the contrary, there were those who felt that the whole question was qualitative rather than quantitative, that it was a question of developing sufficient mathematical maturity to digest the topics desired, that is, a question of growth which requires time. They pointed out that with our present curricula they usually found that students required more time than was allotted them to really understand differentiation, for instance. Attention was called to the fact that many topics omitted from the report were actually bound up theoretically with those selected, hence their deletions might actually increase the students' difficulties.

The attitude of the members of the association was to prepare these points as questions rather than as criticisms. All agreed that we should be receptive in this matter which was primarily the business of the social sciences.

SECOND SESSION OF THE ASSOCIATION

1. "Series of orthogonal polynomials" by Professor Dunham Jackson, University of Minnesota.

2. "Mathematical preparation for a student of modern physical science" by Professor Linus Pauling, California Institute of Technology, by invitation.

3. "On the foundations of commutative algebra" by Professor H. S. Vandiver, University of Texas.

1. Professor Jackson's paper is concerned with systems of polynomials orthogonal over an interval with respect to a given weight function, and in particular with the convergence of expansions in series of such polynomials, giving an account of such properties as are readily deducible by methods comparable with those used in elementary presentations of the theory of Fourier series. Convergence is treated on the basis of the Christoffel-Darboux formula under the assumption that the weight function is such that the polynomials $p_n(x)$ of the normalized orthogonal system are bounded with respect to n for a specified value or range of values of x, and for a broader class of weight functions on the basis of general theorems on approximation by means of polynomials, leading to an estimate of the order of magnitude of the coefficients.

3. Professor Vandiver's paper treats extensions of the theorem in abstract algebra which replaces the theorem in ordinary algebra to the effect that the polynomial $f(x)$ with given rational coefficients vanishes for at least one complex value of x. If $\phi(x)$ is a polynomial with rational coefficients irreducible in the rational field, then residue classes are defined, with respect to a modulus $\phi(x)$, which when combined by the operations addition, subtraction, multiplication and division give a field which is isomorphic with the field generated upon assuming the existence of a quantity α such that $\phi(\alpha)=0$. Indeterminates are explicitly defined and the above mentioned notions extended to algebraic rings.

MEETING OF THE BOARD OF TRUSTEES

A meeting of the board of trustees was held on Monday evening.

The following twenty-four persons were elected to membership on applications duly certified:

S. R. BAKER, A.B.(Ursinus) York, Pa.

JESSIE W. BOYCE, A.M.(Minnesota) Instr., State Teachers Coll., Wayne, Nebr.

B. R. BRICK, B.S.(Minnesota) Instr., Washington Univ., St. Louis, Mo.

J. H. BUTCHART, Ph.D.(Illinois) Asst., Univ. of Illinois, Urbana, Ill.

A. H. CLIFFORD, A.B.(Yale) Student, California Inst. of Tech.

G. S. COOK, A.M.(Kansas) Instr., State School of Mines, Rapid City, S. D.

L. L. CRUISE, M.S.(Kansas State Coll., Fort Hays) Teacher, Jr. Coll., Rochester, Minn.

P. S. DONCHIAN, A.B. (Yale) Vice-President and Genl. Mgr., Donchian Rug Co., Hartford, Conn.

E. A. DRAGO, B. S.(Cooper Union) Mariners Harbor, Staten Island, N. Y.

J. N. EASTHAM, Ph.D.(Catholic Univ.) Head of Dept., Nazareth Coll., Rochester, N. Y.

O. C. EDWARDS, M.E.(Pennsylvania State Coll.) Asst. Prof., Mech. Eng., Genl. Extension Div., Univ. of Minnesota, Minneapolis, Minn.

M. C. ERWIN, Head of Dept., High School, Reynolds, Ind.

R. E. GADSKE, M.S.(Northwestern) Teacher, High School, Carbondale, Ill.

SISTER MARY ALOYSIUS, A.M.(Chicago). Instr., Coll. of St. Catherine, St. Paul, Minn.

SISTER MARY FELICE, Ph.D.(Catholic Univ. of America) Prof., Mount Mary Coll., Milwaukee, Wis.

D. B. PERRY, A.B.(Stanford) Grad. Student, Stanford Univ.; Menlo Park, Calif.

L. J. QUAID, B.S. in E.E.(Illinois) Instr. in Drawing, Univ. of Minnesota, Minneapolis, Minn.

K. E. ROSINGER, Ph.D.(Harvard). Research Fellow, Harvard Univ., Cambridge, Mass.

R. F. SCHNEPP, Ph.D.(Fribourg) Prof. and Head of Dept., St. Mary's Univ. of San Antonio, San Antonio, Texas.

C. N. STOKES, Ph.D.(Minnesota) Asst. Prof., Education; Head of Math. Dept., Univ. High School, Minneapolis, Minn.

A. J. STRANE, Eng. of Mines(Minnesota) Head of Dept., Duluth Jr. Coll., Duluth, Minn.

HELEN E. TERNANDT, B.S.(Northwestern) Chicago, Ill.

BEATRICE M. VELTEN, B.Sc.(New Jersey Coll. for Women) Actuarial Clerk, Metropolitan Life Ins. Co., Rahway, N. J.

E. B. WORTHINGTON, A.M.(Franklin and Marshall) Instr., Franklin and Marshall Academy, Lancaster, Pa.

The trustees voted (1) to authorize the business manager to sign the revised contract for printing the MONTHLY; (2) to approve the addition of J. H. Weaver to the Board of Associate Editors to have charge of advertising matters and also editing the Department of News and Notices; (3) to appoint as a Committee on Arrangements for the summer meeting in Chicago for 1933, Professors H. S.

Everett, E. J. Moulton, Mark Ingraham, and as a Committee on Program, Professors W. D. Cairns, Mayme I. Logsdon and H. W. March, with instructions to report at the Christmas meeting of the Association; (4) to appoint a Committee on Arrangements for the Christmas meeting at Atlantic City, a Committee on Program, and a nominating committee for the coming election.

P. H. DAUS, *Acting Secretary*

THE SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual (Spring) meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Maryland on Saturday, May 7, 1932. About eighty attended.

The following forty-one members were present: Beatrice Aitchison, G. F. Alrich, Ethel M. Anderton, R. M. Ashmun, G. A. Bingley, Archie Blake, C. C. Bramble, W. D. Cairns, Paul Capron, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, P. F. Federico, Michael Goldberg, Patricia Gosnell, Harry Gwinner, W. M. Hamilton, F. E. Johnston, L. M. Kells, A. E. Landry, C. L. Leiper, G. A. Lyle, Florence M. Mears, F. D. Murnaghan, O. J. Ramler, C. H. Rawlins, J. N. Rice, H. M. Robert, Jr., R. E. Root, J. B. Scarborough, Maurice Scheier, J. T. Spann, T. H. Taliaferro, J. H. Taylor, Mildred E. Taylor, F. M. Weida, Paul Wernicke, J. W. Williamson, E. W. Woolard, R. C. Yates, Oscar Zariski. These represented the following institutions: Goucher College, Johns Hopkins University, St. John's College, University of Maryland, U. S. Naval Academy and Postgraduate School, Mary Baldwin College, Catholic University, George Washington University, U. S. Coast and Geodetic Survey, International Boundary Commission, Bureau of Ordnance, U. S. N., U. S. Naval Observatory, U. S. Patent Office, Oberlin College.

The meeting was distinguished by the presence of Professor Richard Courant of Göttingen and Secretary Cairns. The morning session opened with an address of welcome from President Pearson of the University of Maryland and a message of good will and coöperation from Secretary Cairns of the Association. Chairman Woolard presided. The afternoon session opened with an address by Professor T. H. Taliaferro, Dean of the College of the University of Maryland.

At the business meeting it was voted to hold the next meeting of the Section at Johns Hopkins University on the first Saturday of next December. Officers were elected for the ensuing year as follows: Chairman, Paul Capron, U. S. Naval Academy; Secretary, F. M. Weida, George Washington University; other members of the Executive Committee: Tobias Dantzig, University of Maryland and G. T. Whyburn, Johns Hopkins University.

The following papers were presented.

At the morning session:

1. "A second note on the celestial sphere" by Professor Frank Morley, by invitation.

2. "Factorization of a matrix" by Archie Blake.

3. "On the relations of matrices to elementary mathematics" by Professor F. D. Murnaghan, Johns Hopkins University.

4. "A geometric proof of a theorem of Eddington" by Doctor Oscar Zariski.

5. "Projective derivatives" by Professor Tobias Dantzig, University of Maryland.

At the afternoon session:

6. "Difference equations and differential equations" by Professor Richard Courant, University of Göttingen.

7. "Regular accessibility" by Beatrice Aitchison.

8. "Congruences of lines in space" by J. Yerushalmy, by invitation.

Professor Courant's paper was particularly delightful; it gave insight into fundamental characteristics of linear partial differential equations as limiting forms of finite difference equations. The elliptic, hyperbolic, and parabolic types were treated, with simple and convincing illustrations.

Abstracts of other papers follow, numbered in their order on the program:

1. One visualizes a number (the unrestricted or complex number of ordinary algebra) as a point of a plane or sphere, Ω. One may visualize two numbers x and y as the circle orthogonal to Ω at the points x and y. It is labor-saving to attend to only one side of Ω, say the interior, and to speak of x, y as the arc (or if preferred, as the line of a hyperbolic geometry).

A correspondence between x and y gives ∞^2 arcs, which may have an envelope. This envelope we call the caustic of the correspondence, the word caustic being usual in this connection in books on geometrical optics (e.g., J. P. C. Southall, *Geometrical Optics*, Macmillan, 1910). If the correspondence be differentiable, then the arcs x, y and $x+dx$, $y+dy$ have the cross-ratio $dx\,dy/(x-y)^2 = (d\theta)^2$.

The arcs intersect when this is positive, that is, when $d\theta$ is real.

Taking Ω as the plane of the blackboard, then the arc x, y meets a consecutive arc on the normal to the plane at z, where, if $dx/dy = \mu e^{i\phi}$, $z = (x+\mu y)/(1+\mu)$.

Applying the formulas to the two simplest correspondences

$$\alpha xy + \beta x + \gamma y + \delta \tag{1}$$

$$\alpha x\bar{y} + \beta x + \gamma \bar{y} + \delta, \tag{2}$$

we find in the first case that the caustic is a right cone or an inverse of a right cone (a Dupin cyclide), the arcs touching the cone at a constant angle with the generators (or lines of curvature); and in the second case the caustic is a circle or the interior arcs of two circles.

In the Euclidean case when the universe expands and becomes a point, the caustic in (1) becomes a right cylinder. Thus the cylinder which touches three lines at a given angle is the natural extension of the inscribed circle of three lines of a plane.

In the Euclidean case the caustic in (2) is a circle.

2. A proof is given for a theorem of Frobenius, that any square matrix on a finite range can be factored as the product of two symmetric factors, in any field. A restricted extension to Hermitian factors can be made. A new rational construction for the factors is given.

3. In this paper Dr. Murnaghan suggested the exposition of complex number theory in terms of real two-rowed matrices, the matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

corresponding to the complex number $a+ib$, as then the law of multiplication of complex numbers follows naturally from the law of multiplication of matrices. This mode of exposition helps also to remove the confusion between complex numbers and vectors, which is very evident amongst students whose primary interest is electrical engineering and for whom a complex number is essentially an operator which rotates and stretches a vector. Furthermore the method suggested leads at once to the important theory of quaternions (complex two-rowed matrices). De Moivre's theorem appears as

$$e^{\theta S} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \cos\theta . E + \sin\theta . S,$$

$$\text{where } S = i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the unit skew-symmetric matrix. For three-rowed matrices, De Moivre's theorem takes the form $e^{\theta S} = E + \sin\theta + (1-\cos\theta)S^2$, where

$$S = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix}, \quad p^2 + q^2 + r^2 = 1,$$

is the unit skew-symmetric matrix.

This is equivalent to the Rodrigues' parametric expression of three-rowed orthogonal matrices and the method is applicable to orthogonal matrices of any order. The interpretation of a complex number as a two-rowed real matrix of the type mentioned enables one to deduce theorems on canonical forms for real matrices from known theorems on canonical forms for matrices with complex elements.

4. In a recent paper Eddington proves the following theorem: If n complex 4-rowed matrices, E_i, satisfy the two conditions: 1) $E_i^2 = -1$; 2) any two of the given matrices are anticommutative, i.e., $E_iE_j = -E_jE_i$, $i \neq j$, then $n \leqq 5$, and the maximal value 5 is actually reached. If the matrices $E_1, E_2, \cdots, E_5$ are subject to the additional condition to be either real or pure imaginary, then two at

most are real, while the other three are pure imaginary. Eddington's proof is strictly algebraic. The author gives a geometric and a shorter proof of Eddington's theorem, by making use of known facts concerning abelian groups of involutory collineations in space.

With each matrix E_i there is associated a collineation Γ_i in the projective complex space, and *vice versa* Γ_i determines the matrix E_i to within a factor. The condition $E_i^2 = -1$ says that $\Gamma^2 = 1$; that is, Γ_i is involutory, and therefore is either an harmonic homology or an harmonic biaxial collineation. The condition $E_iE_j = -E_jE_i$ says that $\Gamma_i\Gamma_j = \Gamma_j\Gamma_i$. The n collineations Γ_i belong to an abelian group G of involutory collineations. The abelian group cannot be the tetrahedral group, because the matrices of the collineations of this group can be reduced simultaneously to the diagonal form and therefore are not anticommutative. Therefore the group G is the group G_{16} of biaxial harmonic collineations, associated with a quadric surface Q. If Q is given in the canonical form $x_1x_4 - x_2x_3 = 0$, it appears that the matrices E_i are direct products of two two-row matrices, and this representation of the matrices E_i leads immediately to the proof of Eddington's theorem.

7. Definitions with illustrations were given of open and closed sets, of the notions of connectedness and local connectedness, compactness and separability, of simple continuous arcs, of G_δ sets and of accessibility and of regular accessibility. With these as a basis, a very general condition for regular accessibility was stated. A necessary and sufficient condition that a point P be regularly accessible from a connected, locally connected G_δ set G, is that the set $G+P$ be connected and locally connected. Most of the known theorems on accessibility can be shown to be special cases of this proposition.

8. In the representation of the lines of space by the Grassmanian hyperquadric V_4^2 in S_5, the lines of a congruence are given by the points of a surface F on V_4^2. The study of the surfaces of V_4^2 yields results on the congruences of lines in space. Thus the congruences of bisecants to rational space curves are represented by the well known surfaces of Veronese. Translating the known properties of these surfaces into line geometry of space, we obtain the more important properties of these congruences.

PAUL CAPRON, *Secretary*

THE SIXTEENTH ANNUAL MEETING OF THE KENTUCKY SECTION

The sixteenth annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky on Saturday, May 14, 1932. Ill health during the year compelled Professor Smith Park, Eastern State Teachers College, to resign as chairman of the Section on April 6, 1932. Dean P. P. Boyd, University of Kentucky, kindly consented to act as chairman, and members quickly responded to the call for an emergency program.

The attendance was forty-eight including the following twenty members of the Association: N. B. Allison, W. E. Baxter, P. P. Boyd, L. W. Cohen, J. M. Davis, A. R. Fehn, Charles Hatfield, S. F. Hendricks, W. R. Hutcherson, E. R. Keller, C. G. Latimer, Elizabeth LeStourgeon, Buena C. Mathias, W. L. Moore, Sister Charles Mary Morrison, Sallie Pence, D. W. Pugsley, E. L. Rees, Guy Stevenson, H. M. Yarbrough.

The Secretary was instructed to write a letter of deep regret and sympathy to Professor Smith Park. Professor Fehn reported that the Kentucky Section of the Mathematical Association received its "charter" as an affiliated member of the National Council of Teachers of Mathematics, May 1, 1932. The purpose in affiliating with the National Council is to make contact with teachers of secondary and elementary mathematics to further the interests of mathematics in the State. The committee, consisting of P. P. Boyd, chairman, Guy Stevenson and H. M. Yarbrough, to consider ways and means of organizing a branch of the National Council was continued. The following officers were elected for the coming year: Chairman, Charles Hatfield, Georgetown College; Secretary, A. R. Fehn, Centre College.

The following program was presented:

1. "The triangles in-and-circum-scribed to the biflecnodal rational quartic" by Sister Charles Mary Morrison, Nazareth College.

2. "A group of transformations" by Professor C. G. Latimer, University of Kentucky.

3. "A simple problem in mapping" by Professor Guy Stevenson, University of Louisville.

4. "A theorem concerning perfect points in algebraic geometry" by Professor W. R. Hutcherson, Berea College.

5. "The problem of coloring maps" by Professor L. W. Cohen, University of Kentucky.

6. "Problems involving linkages" by Sallie E. Pence, University of Kentucky.

7. "Experiments in laboratory mathematics" by Professor W. L. Moore, University of Louisville.

8. "How to reduce failures in freshman mathematics" by Professor M. C. Brown, University of Kentucky, by invitation.

Abstracts of three of the papers follow:

1. Four possible contours for a biflecnodal rational quartic may be imagined, namely, the form with three crunodes, with three acnodes, with two crunodes; and one crunode. We consider the four mentioned types to discover in each case where the curve proves to be real, the nature of the possible eight triangles. The study shows that

(1) a quartic with three acnodes cannot have a biflecnode.

(2) a rational quartic with a biflecnode cannot have a real in- and circumscribed triangle.

We then depart from the symmetry of the curve and consider a quartic with merely a flecnode and show that a real in- and circumscribed triangle may be obtained.

3. The three distinct points, z_1, z_2, z_3, in the complex z-plane are mapped into the three distinct points, w_1, w_2, w_3, in the w-plane by means of the transformation

$$\frac{(w - w_3)(w_1 - w_2)}{(w - w_1)(w_3 - w_2)} = \frac{(z - z_3)(z_1 - z_2)}{(z - z_1)(z_3 - z_2)},$$

which may be written in the form

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \tag{1}$$

where

$$\begin{aligned}
\alpha &= w_1(z_1 - z_2)(w_2 - w_3) - w_3(w_1 - w_2)(z_2 - z_3),\\
\beta &= z_1w_3(z_2 - z_3)(w_1 - w_2) - w_1z_3(z_1 - z_2)(w_2 - w_3),\\
\gamma &= (z_1 - z_2)(w_2 - w_3) - (w_1 - w_2)(z_2 - z_3),\\
\delta &= z_1(z_2 - z_3)(w_1 - w_2) - z_3(z_1 - z_2)(w_2 - w_3).
\end{aligned}$$

The point $z = g \equiv -\delta/\gamma$ is outside the circle c_1 passing through the points z_1, z_2, z_3, if the order of these points around c_1 is the reverse to that of w_1, w_2, w_3, around the circle c_2 passing through these points, and consequently all points outside c_1 are mapped into points outside c_2, and all points inside c_1 are transformed into points inside c_2. If the order of the points on c_1 is the same as that on c_2 then g is inside c_1 and then all points inside (outside) c_1 are mapped into points outside (inside) c_2 by the transformation (1).

4. Let an algebraic surface be invariant under a cyclic birational transformation T of period n. A point and its images under T constitute an involution. When this involution has only a finite number of fixed points, they are singular points on the surface. The configuration of the surface near each such point is discussed, and it was proved that if $n = 2$, every fixed point is a perfect point. (For a definition of perfect point see W. R. Hutcherson, Maps of certain cyclic involutions on two dimensional carriers, Bulletin of the American Mathematical Society, volume 37, 1931, pp. 759–765.)

A. R. FEHN, *Secretary*

NOTATION OF DECIMAL FRACTIONS IN BOHEMIA

By QUIDO VETTER, Carolina University, Prague

In the year 1923 the late Professor Florian Cajori asked me why it is that in Austria and Bohemia the fractional part of a decimal fraction is separated from the integral part by a raised point, while in other parts of the Continent an ordinary period, or more often a comma, is used. Professor Cajori was then writing his valuable History of Mathematical Notations and his inquiry was

an evidence of the care he took in his work. I am sorry to say, however, that I was at that time prevented by other work from investigating the matter with the care that it deserved, and it is only recently that I have been able to consider it.

Few scientific works on mathematics were written in the Czech regions in the 17th century. Moreover, there was no reason why decimal fractions should get into such arithmetics as appeared, for the measures and weights as well as the coinage were not on a decimal scale, with the exception of those used by surveyors. It is for this reason that Wácslav Joseff Wesselý, in his work *Gruntownj Počátek Mathematického Vměnj* (The Fundamental Beginnings of the Mathematical Art, Prague, 1734, p. 57), in speaking of measures of lengths separates the integers from their "brychy" or "shorter partes" by a period (.) and then places an arc before the right-hand figure, thus: 7.563(3, which stands for 7 rods, 5 feet, 6 inches, 3 grains. It is therefore apparent that the period is not looked upon as a decimal point, but simply as a separatrix of various units. For example, he writes 51.26.42(2 when giving a measure of an angle, meaning what we should express as 51°26′42″.

Father Josef Stěpán Schmidt (1720–1783) gave three ways of writing a decimal fraction. In his *Tabulae mathematicae* (Prague 1757, p. 38) he writes 6.4′.5.″3‴, or 6453‴, or 6,453 to mean what in the United States is expressed by 6.453.

Finally Father Fr. Zeno, a Jesuit, in his *Elementa algebrae, geometriae, trigonometriae* (Prague, 1769, p. 37), writes the decimal fraction 3.879 as 3°8′7″9‴.

A new movement in the scientific life of Prague was developed in the 18th century by certain Liberals who formed a private society later known as Royal Society of Arts. This society published semiannually the Gelehrte Nachrichten, and later the Abhandlungen einer Privatgesellschaft in Böhmen zur Aufnahme der Mathematik, vaterländischen Geschichte und Naturwissenschaften, Abhandlungen der böhmischen Gesellschaft der Wissenschaften in Prag, Abhandlungen der königlichen böhmischen Gesellschaft der Wissenschaften, and various other papers and proceedings and a Gazette.

These publications show that the decimal fraction was indicated by a comma at the end of the 18th century as well as in the first third of the 19th century. Only in the Abhandlungen for 1780, in one of the tables we find the form 5.70, while in another volume (1785) of the same publication, A. Ferkeles uses 0.04347826086. In the second volume of *Philosophiae naturalis principia mathematica*, by Newton, which was published 1785 by the most prominent Czech mathematician of the time, Father Jan Tesánek, S. J. (1728–1788), there occurs in a table (p. 236) the form 0.000001, and a similar usage is seen in a work by Father Stanislav Vydra, S.J., (1741–1804), on p. 96 of his *Počátkově Arytmetyky* (Beginnings of Arithmetics) published by Lad. Jandera in the year 1806. In this work the author gives a systematic treatment of decimal fractions, stating that 6789,2345 may be written 6789.2345, but he does not make any further use of this symbol. He states that decimal fractions were used by sur-

veyors for the reason that their measures were divided decimally. Of the other books dealing with the same subject we may mention two works of Father Lad. Jos. Jandera (1776–1857), a monk in the Strahov monastery, his *Prima calculi exponentialis elementa* (Prague, 1817) where he uses (p. 29) the form 0,6666, and his *Beiträge zu einer leichten und gründlichen Behandlung einiger Lehren der Arithmetik* (Prague, 1830), as on p. 179, where he writes 428,5000000. Mention may also be made of the *Mémoire sur la dispersion de la lumière*, by A. L. Cauchy, published in Prague by the Czech Royal Society in the year 1836, where we find (p. 62) the number 1,330935.

This notation got into the practical life, as we see in a handbook for traders, written by J. Gunz, F. C. Nelkenbrecher's *Taschenbuch der neuesten Münz-Maass- und Gewichtsverfassung aller Länder und Oerter*, ihrer Wechselarten, Usi, Respettage, öffentlichen Banken, Messen und andrer zur Handlung gehörigen Anstalten und Gegenstände (Prague, 1809) where a decimal is written as 30, 48.

At the end of the 18th century in Abhandlungen of the Czech Royal Society the decimal comma is used, as also in the works of Ant. Strnad, S.J., director of the Prague observatory, (1747–1799).

This symbolism continued to be used in the Abhandlungen up to 1850. The decimal comma was adopted by the first head-master of the Technical University in Prague, a well-known scientist, Fr. Jos. Gerstner (1756–1832), as is seen in his *Theorie der Wellen* (1804) and the *Über die ob. Wasserräder*, and by such scholars as Jos. Steinmann (1819), Jos. Jüttner (1833), K. Kreil (1798–1862), the director of the observatory in Prague, K. Fritsch (1812–1897), and Professor Fr. Ad. Petřina (1799–1855). On the other hand, in 1806 another Prague Astronomer, a Premonstrant, Al. David (1757–1836) used the period in his *Längenunterschied zwischen Prag und Breslau*, as in 6.64 (p. 74), but in his other works he adopted the usual continental symbol of a comma.

In the years 1830–1840 the decimal point began to appear again as a period, especially in the works of those authors who had spent some time in the Alpine countries. For example, Jak. Fil. Kulik (1793–1863), a professor at the University in Prague, used the period in his first study published in the Abhandlungen (1832), as did Al. David, Ad. Bittner (1777–1844), Professor Fr. Hyn. Kaj. Hallaschka, and Jos. J. Böhm (1807–1868), who spent some years in Innsbrück as a Professor of mathematics.

The year 1837 marks a new stage in the history of decimal symbols in Bohemia. At that time Professor Christian Doppler, who taught at the technical university in Vienna, was called to Prague. In his first essay, "*Versuch einer analytischen Behandlung etc.*," published in the Abhandlungen (1839) he used the raised period, as the English writers do at present, as in $16 \cdot 66025$. This symbol was popularized by another book, his *Arithmetik und Algebra* (Prague, 1844), in which he explained the use of decimal fractions, using forms like $0 \cdot 543$. His symbolism was adopted by Vil. Matzka (1798–1891), a Professor at the Prague Technical University, in his *Elementarlehre von den Logarithmen* (Prague, 1850), and by many other writers, especially by the authors of the

arithmetics for the elementary schools and of manuals for the secondary schools. Among these authors was Fr. Močník (1814–1892), who in his *Theorie der numerischen Gleichungen* (Vienna, 1839), used the comma, but in the second edition of his *Lehrbuch der Arithmetik für das Untergymnasium* (Vienna, 1851) we find the raised decimal points only, as in $17 \cdot 7482$. His *Počtářství praktické* (Practical Calculation, Prague, 1853) was translated into Czech by Ant. Skřivan, and the translator used the period, although the Italian edition of his algebra for the secondary schools, the *Trattato di algebra pel ginnasio superiore* (Vienna, 1854) by P. Magrini, we find the comma again. In the Czech versions I have found the point above until the appearance of the *Anleitung zum Rechnen für Unter-Realschulen in Kaisertum Oesterreich*, Mit eingeschalteter Terminologie in böhmischer Sprache (Prague, 1858).

In the meantime an important reform of currency was introduced in Austria and together with it the use of the raised point became common. A monetary convention having been concluded between Austria and the other countries of the German Confederation, a new currency was introduced in 1858. In the place of the conventional florin (Konventions-Gulden) which was equal 60 kreutzers, a new florin, equal to 100 kreutzers was introduced. The new money naturally made decimal fractions an essential feature of elementary education, the decimal point being usually the raised period. We find, however, in a booklet printed in 1858, the form 136,5 (5/10 = 50/100) with this note: "The first figure after the comma is a decimal fraction." In Matzka's *Bequemste Tafeln zur wechselweisen Umrechnung des alten und neuen österreichischen Geldes* (Prague, 1858) the author speaks of a "Decimalstrich" and "Decimalpunkt," and writes: "$2 \cdot 78 \cdot 6 = 2$ Neugulden $78\frac{6}{10}$ Neukreutzer." An explanation of the new currency, including the operation with decimals, was published by Močník and was immediately translated into Czech as *Klíc k novému řádu mincovnímu*— (A Key to the New currency, Vienna, 1858) and similar works appeared at about the same time.

Since that time the decimal part of the fraction has been separated from the integral one by a point, usually written above, although occasionally other symbols were given, as in Jos. Soukup's arithmetic in (Prague 1869), where the author remarks, "Decimal fractions are marked off from their units by a point or better by an inverted decimal comma (Decimalpunkt, Decimalstrich)" that is, 555 and 55 hundredths of a florian is given as $555 \cdot 55$ or $555 \cdot 55$.

To summarize the matter we see that at the end of the 18th century and at the beginning of the 19th century with us, as generally on the Continent, the decimals were separated from their integers by a comma below. In the years 1830–1840, the raised decimal point was introduced from Austria; and this symbol, under the influence of the centralized schools and the new currency, continued until the second half of the century.

A LAPLACIAN EQUATION

By E. T. BELL, California Institute of Technology

1. *Introduction.* For $n=2$, Laplace's equation in n dimensions

$$\nabla_n u = 0, \quad \nabla_n \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

has the operator ∇_n completely factorable in the field of complex numbers, so that the general solution for $n=2$ is readily found. For $n=3$, 4, ∇_n is not factorable in a (commutative) field, but is in quaternions; for $n>4$ it is not factorable in any linear associative algebra. Thus the usual method of solving $\nabla_2 u=0$ is incapable of generalization. The well known general solution of Whittaker for $n=3$ depends upon other considerations.

A similar situation in the arithmetic theory of forms suggests a generalization of $\nabla_2 u=0$ other than $\nabla_n u=0$, $n>2$, in which the procedure for $n=2$ goes through for any n. As a special case of such equations has recently received some attention[1] in connection with a certain hypergeometric equation of the third order, originally considered by Clausen,[2] we shall briefly discuss the most general extension of $\nabla_2 u=0$ possible of the type indicated. Humbert's equation is[3]

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 u}{\partial z^3} - 3\frac{\partial^3 u}{\partial x \partial y \partial z} = 0.$$

Although the equations discussed are of no apparent significance in physics (except possibly those of the second order), certain special instances of them, in particular those in which the operator is a norm, as is the case in Humbert's equation, have interesting connections with algebraic number fields.

The equations considered have constant coefficients. No similar discussion for equations with variable coefficients is possible, as the linear differential operators concerned do not then commute.

2. *Canonical form of equation.* In this section the summation convention of the tensor calculus is used: a repeated index, as i in $a_i b_{ij}$, denotes a summation over the values $1, \cdots, n (n>1)$ of i, $a_i b_{ij} = a_1 b_{1j} + \cdots a_n b_{nj}$. The c_{ij} are constants in the field of complex numbers; $|c_{ij}|, \neq 0$, is the determinant whose element in row i, column j is c_{ij}, and c^{ij} denotes the cofactor of c_{ij} divided by $|c_{ij}|$. Hence $c_{ij}c^{ik} = \delta_j{}^k$, where $\delta_j{}^k$ is a Kronecker delta, $\delta_j{}^j = 1$, $\delta_j{}^k = 0$, $j \neq k$.

Let $P(x_1, \cdots, x_n)$ be any homogeneous polynomial of degree n which is completely factorable in the form

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[1] Pierre Humbert, Atti del Congresso Internazionale dei Matematiche, (VI), 1928, vol. 3, pp. 53–57.

[2] Crelle's Journal, vol. 3, p. 89.

[3] The solution of this equation occurs incidentally in Forsyth's Treatise on Differential Equations, p. 449, Ex. 4.

$$P(x_1, \cdots, x_n) = \prod_{j=1}^{n}(c_{ji}x_i).$$

The equation to be discussed is

$$P\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)u = 0. \tag{1}$$

Transforming to new variables $\xi^j (j=1, \cdots, n)$, we write $\xi^j = a^{ji}x_i$, where j, i in ξ^j, a^{ji} are superscripts, and obtain

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial \xi^j}\frac{d\xi^j}{dx_i}, = a^{ji}\frac{\partial}{\partial \xi^j}.$$

Hence, if

$$\frac{\partial}{\partial \xi^j} \equiv c_{ji}\frac{\partial}{\partial x_i},$$

we have

$$\frac{\partial}{\partial \xi^j} = c_{ji}a^{ki}\frac{\partial}{\partial \xi^k}, \quad c_{ji}a^{ki} = \delta_j^k,$$

and therefore

$$c^{il}c_{ji}a^{ki} = c^{il}\delta_j^k, \quad a^{kl} = c^{kl}.$$

Thus (1) is transformed to

$$\frac{\partial^n u}{\partial \xi^1 \cdots \partial \xi^n} = 0, \quad \xi^j \equiv c^{ij}x_i. \tag{2}$$

Denote by ξ_i the vector of $n-1$ components obtained from $(\xi^1, \cdots, \xi^n)$ by suppressing ξ^i. The general solution of (1) is, by (2),

$$u = f_1(\xi_1) + \cdots + f_n(\xi_n), \tag{3}$$

where $f_1, \cdots, f_n$ are arbitrary functions possessing derivatives of the orders demanded by (2).

3. *Particular polynomial solutions.* We shall consider only polynomials in one variable. It is readily seen that each step in what follows can be suitably modified to provide polynomial solutions in more than one variable.

By (3), any constant multiple of $\log \xi^1 + \cdots + \log \xi^n$ is a solution of (1). Hence, and by the linearity of (1), the following are solutions

$$\frac{\partial^s}{\partial x_1^s}V(x_1, \cdots, x_n) \quad (s = 0, 1, 2, \cdots),$$

where

$$V(x_1, \cdots, x_n) \equiv \log Q(x_1, \cdots, x_n), \quad Q(x_1, \cdots, x_n) \equiv \prod_{j=1}^{n}(c^{ij}x_i).$$

Without loss of generality we may assume the coefficient of the highest power of a particular variable, say x_1, to be 1. In what follows we consider V, Q particularly as functions of x_1, and indicate this by writing $V(x_1)$, $Q(x_1)$.

To obtain polynomial solutions connected with an ordinary linear equation, we use a device which Humbert attributes to Giulotto. From the above, the functions

$$V_s(x_1) \equiv \frac{1}{s!}\left[\frac{\partial^s V(x_1 + h)}{\partial h^s}\right]_{h=0} \quad (s = 0, 1, \cdots)$$

are solutions of (1), and

$$V(x_1 + h, x_2, \cdots, x_n) = \sum_{s=0}^{\infty} h^s V_s(x_1, \cdots, x_n).$$

Further, we have

$$Q(x_1 + h) = \sum_{j=0}^{r} \frac{h^j}{j!} Q_j(x_1),$$

where r is the degree of $Q(x_1, \cdots, x_n) \equiv Q(x_1)$ in x_1, and $Q_j(x_1)$ denotes the jth derivative of $Q(x_1)$ with respect to x_1, so that $Q_0(x_1) = Q(x_1)$, $Q_r(x_1) = r!$. The functions $V_s(x_1)$ are now reduced to polynomials in x_1 by constraining $(x_1, \cdots, x_n)$ to an appropriate locus in n-space.

Let $j_1, \cdots, j_a$ be all those values of $j > 0$ for which $Q_j(x_1)$ is not independent of all of $x_2, \cdots, x_n$. Then the locus in question has as its equations

$$Q_0(x_1) = 1, \quad Q_{j_p}(x_i) = 0 \quad (p = 1, \cdots, a).$$

On this locus we have

$$Q(x_1 + h) = 1 + \sum_{j=1}^{r-a-1} h^{s_j} A_{s_j}(x_1) + h^r,$$

where the A's are polynomials in x_1 alone, and the s_j are such that $0 < s_j < r$, $s_i \neq s_j$ $(i \neq j)$. The $V_s(x_1)$ are now polynomials in x_1 alone, say $W_s(x_1)$, and their generating identity is

$$\log (1 + * + h^r) = \sum_{s=0}^{\infty} h^s W_s(x_1),$$

where the * denotes the sum on the right of the expansion for $Q(x_1 + h)$ above.

4. *Remarks.* For Humbert's equation, the polynomials satisfy an equation of hypergeometric type of the third order. By the usual standard processes for obtaining the differential equation satisfied by functions whose generating identity is given, for example as in the case of spherical harmonics or Bessel functions, it is a simple matter to find the differential equation for the $W_s(x_1)$ for any given $P(x_1, \cdots, x_n)$. But it does not seem to be so simple to impose necessary and sufficient restrictions upon the general completely factorable P to give as the final equation one of hypergeometric type. It should be clear from general principles that the device outlined above will lead for Humbert's equation to a hypergeometric equation; possibly it is. But the complexity of the mere algebra in the classical papers (Pochhammer, Goursat, and others) on higher hypergeometric equations seems to preclude any simple general solution.

THE DIRECTION COSINES OF A p-SPACE IN EUCLIDEAN n-SPACE[1]

By STEWART S. CAIRNS, Lehigh University

1. *Analytic definition of the cosines.* Let $(y_1, \cdots y_n)$ be rectangular cartesian coordinates in euclidean n-space. A p-space, L_p, in the n-space may be defined by a set of equations

$$y_i = \sum_{j=1}^{p} a_{ij}u_j + y_i^0 \qquad (i = 1, \cdots, n) \tag{1.1}$$

where the u's are parameters and the matrix $\|a_{ij}\|$ is of rank p. We will refer to $\|a_{ij}\|$ as a *matrix of* L_p. The p-rowed determinants of this matrix will be called *direction components of* L_p and denoted thus:

$$\delta_{i_1 \cdots i_p} = \begin{vmatrix} a_{i_1 1} & \cdots & a_{i_1 p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{i_p 1} & \cdots & a_{i_p p} \end{vmatrix} \qquad (i_1 < i_2 < \cdots < i_p). \tag{1.2}$$

The following quantities will be called the *direction cosines of* L_p:

$$\gamma_{i_1 \cdots i_p} = \frac{\delta_{i_1 \cdots i_p}}{(\sum' \delta^2_{i_1 \cdots i_p})^{1/2}}, \tag{1.3}$$

where $\sum'$ indicates a summation over all the sets $(i_p, \cdots i_p)$. L_p has $n!/[(n-p)!\,p!]$ direction cosines.

2. *A preliminary theorem on matrices.*

Lemma. *In the matrix* $\|a_{ij}\|$ *of* §1, *let*

$$A_{st} = \sum_{i=1}^{n} a_{is}a_{it} \qquad (s = 1, \cdots p;\ t = 1, \cdots p). \tag{2.1}$$

Then

$$\sum' \delta^2_{i_1 \cdots i_p} = \sum'' (-1)^{[i_1 \cdots i_p]} (A_{1 i_1} \cdots A_{p i_p}), \tag{2.2}$$

where $\sum'$ *has the same meaning as in equation* (1.3); $\sum''$ *indicates a summation over all permutations,* $(j_1, \cdots j_p)$, *of* $(1, \cdots p)$, *and* $[j_1, \cdots j_p]$ *denotes the number of inversions in* $(j_1, \cdots j_p)$.

By definition,

$$\sum' \delta^2_{i_1 \cdots i_p} \equiv \sum''' \sum'' (-1)^{[i_1 \cdots i_p]} (a_{i_1 1}a_{i_1 j_1})(a_{i_2 2}a_{i_2 j_2}) \cdots (a_{i_p p})(a_{i_p j_p}), \tag{2.3}$$

where $\sum'''$ indicates a summation over all selections, $(i_1, \cdots i_p)$, of p distinct numbers in any order from the set $(1, \cdots n)$. By equation (2.1),

$$\sum'' (-1)^{[i_1 \cdots i_p]} (A_{1 i_1} \cdots A_{p i_p}) = \sum'' (-1)^{[i_1 \cdots i_p]} \sum^{\text{iv}} (a_{i_1 1}a_{i_1 j_1}) \cdots (a_{i_p p}a_{i_p j_p}), \tag{2.4}$$

[1] Presented to the American Mathematical Society, March 26, 1932.

where $\sum^{iv}$ indicates a summation as the i's run independently through the values $(1, \cdots n)$. The terms of (2.4) for which the i's are all distinct are precisely the terms of the expansion (2.3). Consider a term of (2.4) in which a particular subset of the i's have all the same value. By permuting the corresponding j's, we obtain a number of numerically equal terms, half with plus signs and half with minus signs. Hence the terms of the expansion (2.4) which are not in (2.3) add up to zero. This proves the lemma.

Theorem.[1] *If, in any matrix*

$$\|a_{ij}\| \quad (i = 1, \cdots n; j = 1, \cdots p),$$

we have

$$\sum_{i=1}^{n} a_{is}a_{it} = \begin{cases} 0 \text{ when } s \neq t \\ 1 \text{ when } s = t \end{cases} \quad (s = 1, \cdots p; t = 1, \cdots p), \tag{2.5}$$

then the sum of the squares of all the $n!/[(n-p)!p!]$ *distinct* p*-rowed determinants of* $\|a_{ij}\|$ *is unity.*

This follows at once from the lemma, for, in view of (2.5), the only non-zero term on the right in the expansion (2.2) is $A_{11} \cdots A_{pp} = +1$.

3. *A geometric interpretation of the cosines.*

Theorem. *Employing the notation of* §1, L_p *contains a line in some* $(n-p)$*-space normal to the coördinate* $(y_{i_1} \cdots y_{i_p})$*-space if and only if* $\gamma_{i_1 \cdots i_p} = 0$. *If* $\gamma_{i_1 \cdots i_p} \neq 0$, *let* V *denote a* p*-dimensional volume on* L_p *and* $V_{i_1 \cdots i_p}$ *the volume into which* V *projects on the* $(y_{i_1} \cdots y_{i_p})$*-space. Then*

$$|\gamma_{i_1 \cdots i_p}| = \frac{V_{i_1 \cdots i_p}}{V}. \tag{3.1}$$

The first part of the theorem may be easily verified analytically. For the second part, consider a matrix $\|a_{ij}\|$ of L_p which satisfies the hypothesis of the theorem[2] in §2. This condition on $\|a_{ij}\|$ permits us to regard the u's of equation (1.1) as rectangular cartesian coordinates on L_p with the same unit distance as the y-coordinates. Furthermore, by the theorem of §2, together with equation (1.3),

$$\gamma_{i_1 \cdots i_p} = \delta_{i_1 \cdots i_n}. \tag{3.2}$$

If V denote the volume of a p-dimensional region, R, on L_p, then

$$V = \int_{R'} \cdots \int du_1 \cdots du_p, \tag{3.3}$$

and if $V_{i_1 \cdots i_p}$ denote the volume of the projection, R', of R on the $(y_{i_1} \cdots y_{i_p})$-space, then

[1] For $n = p$, this is a known theorem on determinants. See, for example, Goursat's *Cours d'Analyse*, vol. I (1924), §51.

[2] Such a matrix is henceforth called a *normal matrix of* L_p.

$$V_{i_1 \cdots i_n} = \int_R \cdots \int dy_{i_1} \cdots dy_{i_p}. \tag{3.4}$$

Now a point $(u_1, \cdots u_p)$ on L_p projects into the point on the $(y_{i_1} \cdots y_{i_p})$-space for which

$$y_i = \sum_{j=1}^{p} a_{ij} u_j + y_i^0 \qquad (i = i_1, \cdots, i_p). \tag{3.5}$$

Using equations (3.5) to define a change of variables in the integral of equation (3.4), we find, regarding volumes as positive,

$$\begin{aligned} V_{i_1 \cdots i_p} &= \int_{R'} \cdots \int |\, \delta_{i_1 \cdots i_p} \,|\, du_1 \cdots du_p \\ &= |\, \gamma_{i_1 \cdots i_p} \,|\, V \end{aligned} \tag{3.6}$$

by equations (3.2) and (3.3). Since this result was obtained by means of a special parametric representation of L_p, it should be remarked, to complete the argument, that the direction cosines are independent[1] of the particular parametric representation.

4. *Identities among the direction cosines.* If $\gamma_{1 \cdots p} \neq 0$, a matrix of L_p may be written in the form

$$\left\| \begin{matrix} a_0 & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & a_0 \\ a_{p+1,1} & & \cdots & a_{p+1,p} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n1} & & \cdots & a_{np} \end{matrix} \right\| \tag{4.1}$$

where a_0 has such a value that $\sum' \delta^2_{i_1 \cdots i_p} = 1$ [See §1]. Then, by definition of the direction cosines,

$$\begin{cases} \gamma_{1 \cdots p} = {a_0}^p \\ \gamma_{i_1 \cdots i_{p-k} j_1 \cdots j_k} = (-1)^{[i_1 \cdots i_{p-k} e_1 \cdots e_k]} {a_0}^{p-k} \,|\, a_{st} \,| \end{cases} \tag{4.2}$$

where (1) $(i_1 \cdots i_{p-k})$ is any selection of $(p-k)$ numbers, in order, from $(1, \cdots p)$; (2) $(j_1, \cdots j_k)$ is any selection of k numbers, in order, from $(p+1, \cdots n)$; and (3) $|a_{st}|$ is the kth order determinant of the a's where $s = j_1 \cdots j_k$ and t runs through the complement $(e_1 \cdots e_k)$, in order, of $(i_1 \cdots i_{p-k})$ with respect to $(1, \cdots p)$. In the above, k is any positive integer not greater than either p or $(n-p)$. Letting $k = 1$, we find

[1] At most, a change of parameters can simultaneously change the signs of all the γ's. This follows from the definitions in §1.

$$(4.3) \qquad a_{st} = (-1)^{t-p} a_0^{1-p} \gamma_{1\cdots t-1,t+1\cdots p,s} \qquad (t = 1, \cdots p;\ s = p+1, \cdots n).$$

Hence, substituting in (4.2), we obtain the identities[1]

$$(4.4) \qquad \gamma_{i_1\cdots i_{p-k} j_1\cdots j_k} = (-1)^{[i_1\cdots i_{p-k} e_1\cdots e_k]} \gamma_{1\cdots p}^{1-k} \left| (-1)^{t-p} \gamma'_{ts} \right|,$$

where

$$(4.5) \qquad \gamma'_{ts} = \gamma_{1\cdots t-1,t+1\cdots p,s}$$

and $\left|(-1)^{t-p}\gamma'_{ts}\right|$ is the kth order determinant with the same range of values for t and s as in the determinant of equation (4.2). With the identity

$$(4.6) \qquad \sum{}' \gamma_{i_1^2\cdots i_p} = 1$$

we have a set of $n!/[(n-p)!p!] - p(n-p)$ independent identities among the direction cosines of a p-space in n-space. This is a complete set, for L_p has $n!/[(n-p)!p!]$ direction cosines, and there are $\infty^{p(n-p)}$ p-spaces through a point in n-space. If $\gamma_{1\cdots p} = 0$, a similar set of identities may be found by using, instead of $\gamma_{1\cdots p}$, any non-zero direction cosine of L_p.

We call attention here to the following readily verified facts:

(I) *If* $\gamma_{1\cdots p} \neq 0$, *the direction of* L_p *is uniquely determined by the* $p(n-p)+1$ *direction cosines* $\gamma_{1\cdots p}$; $\gamma_{1\cdots t-1,\ t+1\cdots p,s}$ (*where* $t = 1, \cdots p$; $s = p+1, \cdots n$), *or by the corresponding direction components.* [See (4.2) and (4.3) above.]

(II) *Any set* $\delta_{1\cdots p} \neq 0$; $\delta_{1\cdots t-1,\ t+1\cdots p,s}$ (*where* $t = 1, \cdots p$; $s = p+1, \cdots n$) *of* $p(n-p)+1$ *numbers is a set of independent direction components of some* p-*space,* L_p. *The remaining direction components of* L_p, *and hence its direction cosines, are given by equations* (4.4), *read with* δ *in place of* γ.

5. *Conditions for parallelism and perpendicularity.* To illustrate the use of our direction cosines, we will give a complete discussion of parallelism and perpendicularity.

Our definition of direction cosines, if applied to lines in n-space, coincides with the usual definition. Let $(c_1, \cdots c_n)$ and $(c_1', \cdots c_n')$ be direction cosines of two lines, L_1 and L_1'. Then it is known that L_1 is parallel to L_1' if and only if $c_i = \epsilon c_i'$ $(i = 1, \cdots n)$ where $\epsilon = +1$ or -1, and that L_1 is perpendicular to L_1' if and only if

$$(5.1) \qquad \sum_{i=1}^{n} c_i c_i' = 0.$$

We will call a p-space, L_p, and a q-space, L_q $(q \leqq p)$, *parallel* if every line on L_q is parallel to some line on L_p. We will call them *perpendicular* if each line on either of them is perpendicular to every line on the other. Perpendicularity implies $p+q \leqq n$.

[1] These were found, for determinants, by K. Th. Vahlen, *Ueber die Relationen zwischen den Determinanten einer Matrix*, Crelle's Journal, vol. 112 (1893), pp. 306–310.

(I) *A necessary and sufficient condition*[1] *that* L_p *and* L_q *be parallel* $(q \leqq p)$ *is that some* q *linearly independent lines on* L_q *be parallel to* q *lines on* L_p. *The statement still holds if we restrict the lines on* L_q *to be a system of rectangular axes.*

(II) *A necessary and sufficient condition that* L_p *and* L_q *be perpendicular is that each of* p *linearly independent lines on* L_p *be perpendicular to each of* q *linearly independent lines on* L_q. *The statement still holds if we restrict each set of lines to be a system of rectangular axes.*

(*A*) *Two* p*-spaces,* L_p *and* L_p', *with direction cosines* $(\gamma_{i_1 \cdots i_p})$ *and* $(\gamma'_{i_1 \cdots i_p})$ *respectively, are parallel if and only if*

$$\gamma_{i_1 \cdots i_n} = \epsilon \gamma'_{i_1 \cdots i_p} \tag{5.2}$$

where $\epsilon = +1$ *or* -1.

For, the condition in (I) may in this case be interpreted analytically to mean that some matrix of L_p is also a matrix of L_p' [cf. §1]. The resulting sets of direction cosines are therefore equal. The factor ϵ appears because the signs of all the direction cosines of a p-space can be simultaneously changed by a change of parameters.

(*B*) *A* p*-space,* L_p, *with direction cosines* $(\gamma_{i_1 \cdots i_p})$ *and a* q*-space,* $L_q (q < p)$, *with direction cosines* $(\gamma'_{i_1 \cdots i_q})$ *are parallel if and only if there exists a set* $(c_{i_1 \cdots i_{p-q}})$ *of direction cosines*[2] *such that*

$$\gamma_{i_1 \cdots i_p} = \sum (-1)^{[i_1 \cdots i_{p-q},\, k_1 \cdots k_q]} c_{j_1 \cdots i_{p-q}} \gamma'_{k_1 \cdots k_q} \tag{5.3}$$

where the summation is over all permutations $(j_1, \cdots j_{p-q};\ k_1, \cdots k_q)$ *of* $(i_1, \cdots i_p)$, *with* $(j_1, \cdots j_{p-q})$ *and* $(k_1, \cdots k_q)$ *each in numerical order.*

For, condition (I) is equivalent to the condition that the first q rows of some normal matrix [§3] of L_p should form a normal matrix of L_q. The last $(p-q)$ rows are then a normal matrix of some $(p-q)$-space, L_{p-q}, whose direction cosines $(c_{i_1 \cdots i_{p-q}})$ satisfy (5.3). This becomes clear if one expand the determinant $\delta_{i_1 \cdots i_p} = \gamma_{i_1 \cdots i_p}$ [See (3.2)] by the Laplace development[3] using complementary minors from the first q and the last $(p-q)$ columns.

(*C*) *A* p*-space,* L_p, *with direction cosines* $(\gamma_{i_1 \cdots i_p})$ *is perpendicular to an* $(n-p)$*-space,* L_{n-p}, *with direction cosines* $(\gamma'_{i_1 \cdots i_{n-p}})$ *provided*[4]

$$\gamma_{i_1 \cdots i_p} = \epsilon (-1)^{[i_1 \cdots i_p,\, j_1 \cdots j_{n-p}]} \gamma'_{j_1 \cdots j_{n-p}}, \tag{5.4}$$

where $\epsilon = +1$ *or* -1, *and* $(i_1 \cdots i_p, j_1 \cdots j_{n-p})$ *is a permutation of* $(1, \cdots n)$ *with* $(i_1 \cdots i_p)$ *and* $(j_1 \cdots j_{n-p})$ *each in numerical order.*

We may interpret the condition in (II) above to mean that there exists a

[1] For substantiation of this statement and the following one, see P. H. Schoute, *Mehrdimensionale Geometrie* I, §§2, 3, 6.

[2] That is, a set of numbers satisfying the necessary relationships [cf §4] among the direction cosines of a $(p-q)$-space.

[3] See, for example, H. W. Turnbull, *The theory of determinants, matrices, and invariants* (1929), Ch. II, §5.

[4] Formally the same relationships hold between the homogeneous point and space coordinates of a $(p-1)$-space in $(n-1)$-space. See Weitzenböck, *Invariantentheorie* (1923), pp. 68–72.

normal matrix, $\|a_{i1} \cdots a_{ip}\|$, of L_p and a normal matrix, $\|b_{i1} \cdots b_{i,n-p}\|$, of L_{n-p}, such that the square matrix $\|a_{i1} \cdots a_{ip}\ b_{i1} \cdots b_{i,n-p}\|$ corresponds to a transformation to a new system of rectangular cartesian coordinates in our n-space [See §3]. The determinant of this matrix therefore has the value ± 1. Let it be expanded by the Laplace development using complementary minors of the first p and the last $(n-p)$ columns. This gives us a necessary condition for perpendicularity:

$$\sum(-1)^{[i_1 \cdots i_p j_1 \cdots j_{n-p}]} \gamma_{i_1 \cdots i_p} \gamma'_{j_1 \cdots j_{n-p}} = \pm 1. \tag{5.5}$$

Another condition equivalent to (II) above is that we be able to write matrices of L_p and L_{n-p}, respectively, in the forms

$$\left\| \begin{matrix} 1 & 0 \cdots 0 \\ 0 & 1 \cdots 0 \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ 0 & 0 \cdots 1 \\ A_{11} & \cdots A_{1p} \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ A_{n-p,1} & \cdots A_{n-p,p} \end{matrix} \right\| \quad \text{and} \quad \left\| \begin{matrix} A_{11} & \cdots A_{n-p,1} \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ A_{1p} & \cdots A_{n-p,p} \\ -1 & 0 \cdots 0 \\ 0 & -1 \cdots 0 \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ 0 & 0 \cdots -1 \end{matrix} \right\|. \tag{5.6}$$

For, any two columns, one from each matrix, are direction components of two perpendicular lines, one on L_p, one on L_{n-p} [cf (5.1)]. The above matrix of L_p assumes only that $\gamma_{1 \cdots p} \neq 0$, which involves no loss of generality. From (5.6) we could determine the direction components $(\delta_{i_1 \cdots i_p})$ of L_p in terms of those of L_{n-p}, and hence deduce necessary and sufficient conditions for perpendicularity. It is somewhat simpler, however, to note from (5.6) the numerical equality $|\delta_{i_1 \cdots i_p}| = |\delta'_{j_1 \cdots j_p}|$ [See (C) for the subscripts]. Hence

$$|\gamma_{i_1 \cdots i_p}| = |\gamma'_{j_1 \cdots j_{n-p}}|. \tag{5.7}$$

The correct sign may now be determined by comparing equations (5.5) and (4.6). This establishes condition (5.4).

(*D*) *A q-space, L_q, with direction cosines $(\gamma'_{i_1 \cdots i_q})$ is perpendicular to an $(n-p)$-space, L_{n-p} $(q<p)$, with direction cosines $(\gamma_{i_1 \cdots i_{n-p}})$ if and only if there exists a set $(c_{i_1 \cdots i_{p-q}})$ of direction cosines such that*

$$(-1)^{[m_1 \cdots m_{n-p}, i_1 \ldots i_p]} \gamma_{m_1 \cdots m_{n-p}} = \sum(-1)^{[j_1 \cdots j_{p-q}\ k_1 \cdots k_q]} c_{j_1 \cdots j_{p-q}} \gamma'_{k_1 \cdots k_q} \tag{5.8}$$

where (1) *$(m_1 \cdots m_{n-p}, i_1 \cdots i_p)$ is any permutation of $(1, \cdots n)$* with *$(m_1 \cdots n_{n-p})$ and $(i_1 \cdots i_p)$ both in numerical order and* (2) *the remaining notation has precisely the same meaning as in equation* (5.3).

For, let L_p be perpendicular to L_{n-p}. Then L_q is parallel to L_p. Our conclusion now follows from (B) and (C).

One can similarly deduce from (B) and (C) conditions that a given p-space, L_p, and a given q-space, L_q, contain, respectively, a pair of parallel (or perpendicular) spaces, L_r and L_s, thus completing a discussion of the various possibilities with respect to perpendicularity and parallelism.

CHARACTERISTIC PROPERTIES OF THE EUCLIDEAN LENGTH INTEGRAL

By LINCOLN LA PAZ, Ohio State University

For a non-singular problem of minimizing an integral

$$I = \int_{x_1}^{x_2} f(x, y, z, y', z')dx \tag{1}$$

a direction $dx:dy:dz$ is said to be *transversal* to the extremal through the point (x, y, z) in the direction $1:y':z'$ in case

$$(f - y'f_{y'} - z'f_{z'})dx + f_{y'}dy + f_{z'}dz = 0. \tag{2}$$

In (2) the arguments in f and its partial derivatives are (x, y, z, y', z'). If, in particular, $f \equiv (1+y'^2+z'^2)^{1/2}$ the extremals are straight lines and the transversality condition (2) reduces to

$$dx + y'dy + z'dz = 0, \tag{3}$$

so that a direction transversal to an extremal is orthogonal to it.

Conversely, by employing the lemma that the most general integral (1) for which transversality is equivalent to orthogonality has an integrand function of the special form $f(x, y, z, y', z') = g(x, y, z)(1+y'^2+z'^2)^{1/2}$, it can be shown that the Euclidean length integral is the only one for which transversality is orthogonality and the extremals are straight lines.[1] It is the purpose of the present note to give a proof of this converse theorem which is independent of the lemma just referred to and which illustrates a method believed to be of interest.

The most general integrand function f of a problem of mimimizing the integral (1) for which the extremals are straight lines was first found by Hamel[2] but has been determined in more satisfactory fashion in a paper recently published by D. R. Davis[3] who shows that f must have the form

$$f = g(x, y, z, y', z') + (d/dx)t(x, y, z), \tag{4}$$

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[1] For a proof of the corresponding lemma and theorem in space of $(n+1)$-dimensions see the author's papers *Problems of the Calculus of variations with prescribed transversality conditions*, Bulletin of the American Mathematical Society, vol. 36 (1930), p. 680 and *The Euler equations of problems of the calculus of variations with prescribed transversality conditions*, Proceedings of the National Academy of Sciences, vol. 17 (1931), p. 461.

[2] Hamel, G., *Über die Geometrieen, in denen die Geraden die Kürzesten sind*, Mathematische Annalen, vol. 57 (1903), p. 255.

[3] Davis, D. R., *The inverse problem of the calculus of variations in higher space*, Transactions of the American Mathematical Society, vol. 30 (1928), p. 724.

where t is an *arbitrary* function of (x, y, z) and where g is a *particular* solution of the system

$$(5)\qquad g_{y'y'} = P(y', z', u, v), \quad g_{y'z'} = Q(y', z', u, v), \quad g_{z'z'} = R(y', z', u, v),$$
$$u = y - y'x, \; v = z - z'x,$$

in which $PR - Q^2 \neq 0$ and

$$(6)\qquad P_z = Q_y, \; Q_z = R_y; \; P_{z'} = Q_{y'}, \; Q_{z'} = R_{y'},$$

but P, Q and R are otherwise *arbitrary* functions of their arguments.

For an integrand of the form (4) the transversality condition (2) becomes

$$(7)\qquad (g - y'g_{y'} - z'g_{z'} + t_x)dx + (g_{y'} + t_y)dy + (g_{z'} + t_z)dz = 0.$$

Hence if transversality is to be equivalent to orthogonality the equations (7) and (3) must define the same plane of directions $(dx:dy:dz)$ for all sets (x, y, z, y', z') in the fundamental region R of the integral (1). Hence g and t must satisfy relations of the form

$$(8)\qquad \begin{aligned} g - y'g_{y'} - z'g_{z'} + t_x &= h, \\ g_{y'} + t_y &= hy', \\ g_{z'} + t_z &= hz', \end{aligned}$$

in which $h \neq 0$ is a function of (x, y, z, y', z').

If h is eliminated between these equations it is found that g and t must satisfy the relations

$$(9)\qquad \begin{aligned} g_{y'} + t_y &= y'(g - y'g_{y'} - z'g_{z'} + t_x), \\ g_{z'} + t_z &= z'(g - y'g_{y'} - z'g_{z'} + t_x). \end{aligned}$$

If (9_1) is differentiated partially with respect to y' and (9_2) partially with respect to z' it is possible to show from the resulting equations that $(1+y'^2)P = (1+z'^2)R$. Similarly by differentiating (9_1) with respect to z' and (9_2) with respect to y' the second and third relations in (10) below are obtained.

$$(10)\qquad \begin{aligned} (1 + y'^2)P \qquad\qquad - (1 + z'^2)R &= 0, \\ y'z'P + (1 + z'^2)Q \qquad\quad &= 0, \\ (1 + y'^2)Q + \qquad y'z'R &= 0. \end{aligned}$$

Since the rank of the matrix of coefficients of the system (10) is found to be two, this system admits a single solution (p, q, r) to which all other solutions (P, Q, R) are proportional. An obvious solution of (10) is:

$$(11)\qquad p = (1 + z'^2), \quad q = -y'z', \quad r = (1 + y'^2).$$

Hence the most general solution of (10) for $P(y', z', u, v)$, $Q(y', z', u, v)$ and $R(y', z', u, v)$ is

$$P = k(1 + z'^2), \quad Q = -ky'z', \quad R = k(1 + y'^2), \tag{12}$$

where k is an arbitrary function of y', z', u, v.

However, it can be shown that if the functions P, Q, R in (12) are to satisfy the first condition (6) then k is free of u and v, for the relations $P_z = Q_y$, $Q_z = R_y$ require

$$\begin{aligned} (1 + z'^2)k_v &= -y'z'k_u, \\ -y'z'k_v &= (1 + y'^2)k_u, \end{aligned} \tag{13}$$

and these equations are seen to imply that $k_u = k_v = 0$ since the determinant of the coefficients

$$\delta = \begin{vmatrix} 1 + y'^2 & y'z' \\ y'z' & 1 + z'^2 \end{vmatrix} \tag{14}$$

is different from zero.

The form of $k(y', z')$ has yet to be determined so that the solutions (12) satisfy also the second set of relations $P_z' = Q_y'$, $Q_z' = R_y'$ in (6). Imposing these conditions on (12) we find that k must satisfy the partial differential equations

$$\begin{aligned} (1 + y'^2)\partial k/\partial y' + y'z'\partial k/\partial z' + 3ky' &= 0, \\ y'z'\partial k/\partial y' + (1 + z'^2)\partial k/\partial z' + 3kz' &= 0. \end{aligned} \tag{15}$$

To integrate (15) transform this system into a homogeneous system in the first partial derivatives of a function $G(y', z', k)$ with $\partial G/\partial k \neq 0$ which defines k by means of the relation $G = \text{constant}$. The resulting system

$$\begin{aligned} U_1G &= (1 + y'^2)\partial G/\partial y' + y'z'\partial G/\partial z' - 3ky'\partial G/\partial k = 0, \\ U_2G &= y'z'\partial G/\partial y' + (1 + z'^2)\partial G/\partial z' - 3kz'\partial G/\partial k = 0, \end{aligned} \tag{16}$$

is a complete system, for the equations (16) are independent since the matrix of coefficients contains the second order determinant δ which is different from zero; and, moreover, the commutator

$$(U_1U_2)G = z'\partial G/\partial y' - y'\partial G/\partial z' \tag{17}$$

is seen to be a linear combination of U_1G and U_2G.

A particular solution of (16) is

$$G = k(1 + y'^2 + z'^2)^{3/2}. \tag{18}$$

Hence, since the general integral of a complete system of two equations in three independent variables is an arbitrary function of a single particular integral of

the system,[1] we readily find from the general solution of (16) that the most general non-singular solution k of equations (15) is

$$k = m(1 + y'^2 + z'^2)^{-3/2}, \tag{19}$$

where m is a constant. Therefore the most general solution (P, Q, R) of (10) which satisfies all of the conditions (6) is (12), where k has the value (19) with $m \neq 0$.

In view of equations (5) it is now seen that a particular solution for the function g is

$$g = m(1 + y'^2 + z'^2)^{1/2}. \tag{20}$$

The arbitrary function t in (4) must now be so determined that t and the function g in (20) satisfy (9). On substituting for g from (20) the system (9) reduces to

$$t_y = y' t_x, \quad t_z = z' t_x. \tag{21}$$

Hence, since t is a function of x, y and z alone,

$$t_x = t_y = t_z = 0, \tag{22}$$

and consequently the most general solution for the integrand function f in (4) is given by

$$f = m(1 + y'^2 + z'^2)^{1/2}, \quad m \neq 0. \tag{23}$$

Since two problems (1) for which the integrand functions differ only by a constant factor m are not regarded as distinct it is seen that the Euclidean length integral gives the only problem (1) for which transversality is orthogonality and the extremals are straight lines.

A CLASSIFICATION OF QUADRICS IN AFFINE n-SPACE BY MEANS OF ARITHMETIC INVARIANTS

By RICHARD S. BURINGTON, Case School of Applied Science

1. *Introduction.* The classification of real conics under the euclidean group has been discussed in this Monthly by MacDuffee,[2] Paradiso,[3] and Franklin,[4] with the use of algebraic invariants.

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[1] Goursat, *Leçons sur l'Intégration des Équations aux Dérivées Partielles du Premier Ordre* (1921), p. 71.

[2] C. C. MacDuffee, *Euclidean Invariants of Plane Second Degree Curves*, American Mathematical Monthly, Vol. 33, (1926), pp. 243–252.

[3] L. J. Paradiso, *A Classification of Second Degree Loci in Space*, American Mathematical Monthly, Vol. 33, (1926), pp. 406–418.

[4] P. Franklin, *The Classification of Quadrics in Euclidean n-space by Means of Covariants*, American Mathematical Monthly, Vol. 34, (1927), pp. 453–467.

In some of their euclidean canonical forms, there are infinitely many quadrics, all of which are equivalent under the affine group, and not equivalent to a quadric of any other canonical form. Thus the problem of the separation into types is a problem in real affine geometry.

It is the purpose of this paper to show that the matrix A of the quadric has four arithmetic invariants under the real affine group which are sufficient to give a complete separation of the quadrics into types. These invariants are the ranks and signatures R_1, S_1, R_2, S_2, of A and B, where B is the matrix A with the last row and column deleted. The types obtained by this means coincide exactly with those previously obtained, and the labor[4] in applying the theory to a given quadric is slight. The types of real quadrics for $n=2$, 3, and 4 are listed in detail.

2. *Invariants.* We consider the symmetric matrix

$$A \equiv (a_{ij}),\ i, j = 1, \cdots, n+1, \tag{2.1}$$

of the real quadric

$$F = \sum_{j=1}^{n+1} a_{ij}x_ix_j,\quad x_{n+1} = 1, \tag{2.2}$$

under the real affine non-singular transformations

$$x_i = \sum_{j=1}^{n+1} b_{ij}x_j',\ i = 1, \cdots, n+1,\ b_{n+1,i} = \delta_{i,n+1}\cdot b_{n+1,n+1}, \tag{2.3}$$

whose matrix is

$$T \equiv \begin{pmatrix} b_{11} \cdots b_{1n}\ b_{1,n+1} \\ \cdot \\ \cdot \\ b_{n1} \cdots b_{nn}\ b_{n,n+1} \\ 0 \quad \cdots 0 \quad b_{n+1,n+1} \end{pmatrix},\ d(T) \neq 0, \tag{2.4}$$

where $\delta_{i,n+1}=1$ if $i=n+1$, $\delta_{i,n+1}=0$ if $i\neq n+1$, and $d(T)$ is the determinant of T.

Under T, A becomes

$$\overline{A} = T'\cdot A\cdot T, \tag{2.5}$$

where T' is the transpose of T. If $\overline{B}$ is $\overline{A}$ with the last row and column deleted, then

$$\overline{B} = S'\cdot B\cdot S, \tag{2.6}$$

where S is T with the last row and column deleted and B is A with the last row and column deleted. Thus B is an *invariant matrix* of A, for we can get $\overline{B}$ either

[4] For easy methods of obtaining S_1 and S_2, see Bôcher, *Higher Algebra*, pp. 146–147; or Dickson, *Modern Algebraic Theories*, p. 88.

(i) by transforming and then deleting the last row and column, or (ii) by deleting the last row and column of A and T and then transforming.

Let the *ranks* and *signatures*[1] of A and B be denoted by R_1, S_1, and R_2, S_2, respectively. If A can not be reduced by a T to a diagonal matrix,[2] then S_1 is meaningless.

As is well known, R_1, S_1; R_2, S_2, *are arithmetic invariants of* A *and* B *under* T, *and hence of* A *and* F.

3. *Reduction to Canonical Forms.* We assume that F is of second degree, i.e., $R_2 \neq 0$. From the theory of quadratic forms,[3] we know that there exists a T such that

$$\overline{A} = T' \cdot A \cdot T = \begin{pmatrix} E & \phi \\ \theta & H \end{pmatrix}, \tag{3.1}$$

where $E \equiv (a_{ij})$ is a diagonal matrix with $a_{ij} = 0$, $i \neq j$; $a_{ii} = 1$ for $i = 1, \cdots, P_2$ where $2P_2 = (R_2 + S_2)$; $a_{ii} = -1$ for $i = (P_2+1), \cdots, R_2$; $a_{ii} = 0$ for $i = (R_2+1), \cdots, (n-1)$; ϕ and θ are rectangular matrices all of whose elements are zero, with $(n-1)$ rows and 2 rows, respectively, and

$$H \equiv \begin{pmatrix} a_{nn} & a_{n,n+1} \\ a_{n+1,n} & a_{n+1,n+1} \end{pmatrix}.$$

Suppose $R_1 = r+1$ or r, $R_2 = r > 0$. If $a_{nn} \neq 0$, the transformation[4] (2.4) with $b_{ij} = \delta_{ij}$, except $b_{n,n+1} = -a_{n,n+1}/a_{nn}$ reduces A to a diagonal matrix which can be written

$$A \equiv (a_{jj}), \quad R_2 = r, \quad R_1 = r+1 \text{ or } r, \tag{3.2}$$

where (a_{jj}) is the $E \equiv (a_{ii})$ of (3.1) with R_1, S_1, $n+2$ substituted for R_2, S_2, and n. If $a_{nn} = 0$, interchange a_{nn} and a non-zero element a_{kk} of E and proceed as before.

If $R_2 = r = R_1 - 2$, the transformation T with $b_{ij} = \delta_{ij}$, except $b_{nn} = 1/a_{n,n+1}$, $b_{n,n+1} = -a_{n+1,n+1}/2a_{n,n+1}$ reduces (3.1) to the form

$$A \equiv \begin{pmatrix} E & \phi \\ \theta & P \end{pmatrix}, \quad P \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_2 = r = R_1 - 2. \tag{3.3}$$

It is easy to show that transformation (2.4) is insufficient to reduce (3.3) to form (3.2). Such a matrix (3.3) is called *parabolic.*

From (3.1) we note $0 \leqq R_1 - R_2 \leqq 2$, $0 \leqq |S_1 - S_2| \leqq 1$. Also, we see

Theorem I. A necessary and sufficient condition that A *be parabolic is* $R_1 - R_2 = 2$.

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[1] See Bôcher, loc. cit., p. 146.

[2] Dickson, *Algebras and their Arithmetics*, p. 173.

[3] See Bôcher, *Higher Algebra*, p. 146 and pp. 171–3; and also Dickson, *Modern Algebraic Theories*, p. 72; for notation in (3.1) see pp. 251–2.

[4] Bieberbach-Bauer, *Algebra*, pp. 87–8.

The following theorems establish the fact that the invariants R_1, R_2, S_1, S_2 form a *complete system* of arithemetic invariants. We have proved

Theorem II. The matrix A of F can be reduced by a real non-singular affine linear transformation to a diagonal canonical form (3.2) *for which*

$$F_1 = x_1^2 + x_2^2 \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_q^2, \text{ if } R_1 - R_2 \neq 2,$$

and to a canonical form (3.3) *for which*

$$F = F_1 + 2x_n \cdot x_{n+1}, \text{ if } R_1 - R_2 = 2.$$

We state the fundamental[1]

Theorem III. A necessary and sufficient condition that two forms F_1 and F_2 be equivalent with regard to transformation (2.3) *is that A_1 and A_2 have the same ranks and signatures R_1, R_2, S_1, S_2.*

In view of these theorems, *we classify the quadrics F according to R_1, R_2, S_1, S_2, as indicated in Table* 1. The signatures occuring in the table are positive or zero. A similar table could be built for S_1 and S_2 negative.

4. *Application to the locus $F=0$.* It is evident that this mode of classification permits us some choice in our canonical forms for the locus $F=0$. For example, $x^2-y^2-1=0$, $y^2-x^2-1=0$, $x^2-y^2+1=0$, with corresponding diagonal matrices (written horizontally) $(1\ -1\ -1)$, $(-1\ 1\ -1)$, $(1\ -1\ 1)$, could be canonical forms for the hyperbola. The possible number of canonical forms for $F=0$ and A is finite.

The loci $F=0$ and $(-F)=0$ are equivalent. But the signatures for F are the negatives of those for $(-F)$. Hence, in dealing with a classification of loci $F=0$, we consider only the numerical values of the signatures.

We divide the loci $F=0$ according to R_1, R_2, $|S_1|$, $|S_2|$, as indicated in Table 1.

In the selection of canonical forms we agree to use (3.2) and (3.3) with the understanding that $a_{11}=1$, (if F makes $a_{11}=-1$ use $-F$), and that $a_{n+1,n+1}=-1$ if $|S_1|$ leaves a choice in the matter.

With $F=0$, theorems II and III hold with S_1 and S_2 replaced by $|S_1|$ and $|S_2|$ respectively.

5. Tables 2, 3 and 4 show the classification for $n=2, 3, 4$.

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[1] Bôcher, *Higher Algebra*, p. 148, Theorem 2.

Table 1. CLASSIFICATION OF QUADRICS IN N-SPACE

R_1	R_2	S_1	S_2	R_1	R_2	S_1	S_2
$n+1$	n	$n+1$	n	r	r	r	r
		$n-1$	n			$r-2$	$r-2$
		$n-1$	$n-2$			.	.
		$n-3$	$n-2$			.	.
		.	.			.	.
		.	.				
		.	.	r	$r-1$	r	$r-1$
		$n-k+1$	$n-k$			$r-2$	$r-1$
		$n-k-1$	$n-k$			$r-2$	$r-3$
		.	.			.	.
		.	.			.	.
		.	.			.	.
$n+1$	$n-1$		$n-1$	r	$r-2$		$r-2$
			$n-3$				$r-4$
			.	.	.	.	.
			.	.	.	.	.
			.	.	.	.	.
n	n	n	n	.	.	.	.
				.	.	.	.
		$n-2$	$n-2$	1	1	1	1
		.	.				
		.	.				
		.	.				
.	.	.	.				
.	.	.	.				
.	.	.	.				

Table 2, $n=2$

R_1	R_2	$\lvert S_1\rvert$	$\lvert S_2\rvert$	*Canonical Form*	*Name*
3	2	3	2	(1 1 1)	imag. ellipse
3	2	1	2	(1 1 −1)	ellipse
3	2	1	0	(1 −1 −1)	hyperbola
3	1		1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	parabola
2	2	2	2	(1 1 0)	point
2	2	0	0	(1 −1 0)	pair intersecting lines
2	1	2	1	(1 0 1)	imag. parallel lines
2	1	0	1	(1 0 −1)	real parallel lines
1	1	1	1	(1 0 0)	coincident lines

Table 3, $n=3$

R_1	R_2	$\lvert S_1\rvert$	$\lvert S_2\rvert$	*Canonical Form*	*Name*
4	3	4	3	(1 1 1 1)	imaginary ellipsoid
4	3	2	3	(1 1 1 −1)	real ellipsoid
4	3	2	1	(1 −1 −1 −1)	hyperboloid of two sheets
4	3	0	1	(1 1 −1 −1)	hyperboloid of one sheet
3	3	3	3	(1 1 1 0)	point locus (imag. cone)
3	3	1	1	(1 1 −1 0)	real cone
4	2		2	$\begin{pmatrix} 1 & 1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$	elliptic paraboloid
4	2		0	$\begin{pmatrix} 1 & -1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$	hyperbolic paraboloid
3	2	3	2	(1 1 0 1)	imaginary cylinder
3	2	1	2	(1 1 0 −1)	elliptic cylinder
3	2	1	0	(1 −1 0 −1)	hyperbolic cylinder
3	1		1	$\begin{pmatrix} 1 & 0 & 0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$	parabolic cylinder
2	2	2	2	(1 1 0 0)	single line
2	2	0	0	(1 −1 0 0)	intersecting planes
2	1	2	1	(1 0 0 1)	two imag. parallel planes
2	1	0	1	(1 0 0 −1)	two real parallel planes
1	1	1	1	(1 0 0 0)	two coincident planes

Table 4, $n=4$

R_1	R_2	$\lvert S_1\rvert$	$\lvert S_2\rvert$	*Can. Form*	R_1	R_2	$\lvert S_1\rvert$	$\lvert S_2\rvert$	*Can. Form*
5	4	5	4	(1 1 1 1 1)	4	4	4	4	(1 1 1 1 0)
5	4	3	4	(1 1 1 1 −1)	4	4	2	2	(1 1 1 −1 0)
5	4	3	2	(1 −1 −1 −1 −1)	4	4	0	0	(1 1 −1 −1 0)
5	4	1	2	(1 1 1 −1 −1)	4	3	4	3	(1 1 1 0 1)
5	4	1	0	(1 −1 1 −1 −1)	4	3	2	3	(1 1 1 0 −1)
5	3		3	$\begin{pmatrix} 1 & 1 & 1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$	3	2	1	2	(1 1 0 0 −1)
5	3		1	$\begin{pmatrix} 1 & 1 & -1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$	3	2	1	0	(1 −1 0 0 1)
4	3	2	1	(1 1 −1 0 1)	3	1		1	$\begin{pmatrix} 1 & 0 & 0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$
4	3	0	1	(1 1 −1 0 −1)	2	2	2	2	(1 1 0 0 0)
4	2		2	$\begin{pmatrix} 1 & 1 & 0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$	2	2	0	0	(1 −1 0 0 0)
4	2		0	$\begin{pmatrix} 1 & -1 & 0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$	2	1	2	1	(1 0 0 0 1)
3	3	3	3	(1 1 1 0 0)	2	1	0	1	(1 0 0 0 −1)
3	3	1	1	(1 1 −1 0 0)	1	1	1	1	(1 0 0 0 0)
3	2	3	2	(1 1 0 0 1)					

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

A SQUARE ROOT METHOD AND CONTINUED FRACTIONS

By RAYMOND GARVER, University of California at Los Angeles.

A number of writers have studied a rather interesting method for approximating square roots which may be described briefly as follows: If a is an approximation to the square root of N, then $(a+N/a)/2$ is a better approximation. It was used by Heron of Alexandria about 200 A.D., and by several writers in the middle ages, but the first complete investigation of the method seems to be due to Bouton.[1] James[2] studied and generalized the device, in 1924, and mentioned that it was equivalent to the old approximation, $(a^2+b)^{\frac{1}{2}}=a+b/(2a)$, known as early as the tenth century. Both Bouton and James gave upper limits to the error committed in stopping after a certain number of approximations. Very recently Boys[3] has again described the method. In his paper he keeps the successive approximations in fractional form (assuming N is an integer or fraction), whereas the earlier writers seem to have reduced them to decimals.

I should like to indicate a relation between this method and the theory of continued fractions. This relation seems to make it possible to employ the method to best advantage, and gives at once a very simple upper limit to the error of any approximation.

The theorem in continued fractions which applies is the following, which may be found in Chrystal's Algebra,[4]

If c is the number of partial quotients in the cycle of the continued fraction which represents $\sqrt{N}/M$, p_n/q_n is the nth convergent to the continued fraction, and m is any positive integer, then

$$\frac{p_{2mc}}{q_{2mc}} = \frac{p_{mc}^2 + (N/M^2)q_{mc}^2}{2p_{mc}q_{mc}}.$$

For the moment take $M=1$. If our approximation to $\sqrt{N}$ is the rational fraction a/b, the iterated, or second, approximation is $(a/b+Nb/a)/2$, which reduces to $(a^2+Nb^2)/2ab$. This shows that if a/b is a convergent of the form p_{mc}/q_{mc}, the iterative method steps ahead to another convergent twice as far out in the series of convergents, and so on. At each step we have the best possible rational approximation p_n/q_n of all fractions having denominators not greater than q_n, and we know at once that p_n/q_n is not in error by more than $1/q_n^2$

[1] Annals of Mathematics, ser. 2, vol. 10, 1909, 167–172.

[2] American Mathematical Monthly, vol. 31, 1924, 471–475.

[3] Mathematical Gazette, vol. 16, 1932, 111–115.

[4] Part 2, 2nd ed., 1926, page 468.

and, in fact, by not more than $1/(a_{n+1}q_n^2)$, where a_{n+1} is the $(n+1)$st partial quotient. Further, we can tell immediately whether p_n/q_n is too large or too small.

If we have $c=1$, which is the case when N is one more than a perfect square we may use any convergent as our first approximation. Thus

$$\sqrt{2} = 1 + \frac{1}{2} + \frac{1}{2} + \cdots,$$

and two iterations on 3/2 give 577/408, which is too large (since it is an even convergent) by an amount not greater than $1/2\cdot 408^2$, or about .000003. Peculiarly enough, an iteration on 4/3, which is not a convergent, gives 17/12, which is. This is not usually the case.

When $c=2$, which holds when N is an integer one less than a perfect square, or of the form A^2+A, our theorem would seem to indicate that we should start with an even convergent. It still is true, however, that any convergent iterates to give the convergent twice as far out in the series, though I shall not prove this. Thus, in approximating

$$\sqrt{3} = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{2} + \cdots,$$

a single iteration on the fifth convergent, 19/11, gives the tenth convergent, 362/209, which can not be too large by more than 1 1/4 in the fifth decimal place. There is not, however, any advantage in starting with an odd convergent; the second convergent, 2, is perfectly satisfactory as a first approximation, though some writers have stressed the desirability of getting a close first approximation. Without the aid of continued fractions, a person might take 17/10 as an easily found and close approximation; an iteration on it gives 589/340. This is not, however, as close to $\sqrt{3}$ as is 97/56, the eighth convergent, or the result of interating twice on 2.

If c is greater than 2, the continued fraction expansion may not be easily found. If it is, the method of this note may well be used. Thus,

$$\sqrt{7} = 2 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{4} + \cdots.$$

That is, $c=4$, and it would be desirable to take the fourth convergent, 8/3, as the first approximation. One iteration gives 127/48, which is too large by not more than 1 in the fourth decimal place. For values of c greater than 2, some convergents which are not of the form p_{mc}/q_{mc} may iterate to give convergents, and some non-convergents iterate to convergents. It does not seem easy to obtain general results covering these situations.

The approximation of the square root of a fraction requires no further discussion, since Chrystal's discussion and theorem cover this case. The only reason for writing the fraction in the form N/M^2 is that it makes the continued fraction expansion easier to obtain. In the actual iterations this form need not be used.

Note, added in proof. The usual upper limit for the error, $1/q_nq_{n+1}$, is not used in this paper, since it would usually not be easy to set up. It is possible, however by using another property of continued fractions to obtain a better upper limit than the one used here. For, one or more iterations on a convergent p_{mc}/q_{mc} will give a convergent of the form p_{2sc}/q_{2sc}, s an integer. And it is well known that $p_{2sc}^2 - Nq_{2sc}^2 = 1$. Now, writing p_{2sc}/q_{2sc} without subscripts for convenience, we have the error $E = p/q - \sqrt{N} = 1/q(p+q\sqrt{N})$. Since p/q is an even convergent it is greater than $\sqrt{N}$. Replacing p by $q\sqrt{N}$ in the expression for E then gives an upper limit $1/2q^2\sqrt{N}$ for the error. To make this suitable for computation $\sqrt{N}$ may be replaced by any available approximation less than $\sqrt{N}$; Nq/p is a satisfactory approximation of this type. For this value the upper limit is equivalent to Bouton's, but it is in a simpler form; if a better approximation is available, a better upper limit will result.

A very simple lower limit for E can be found by replacing $q\sqrt{N}$ by p, which is larger than it. E is at once seen to be greater than $1/2pq$, which is equivalent to Bouton's lower limit but is considerably easier to use.

These two limits may be used together very advantageously.

On Equilateral Triangles

By R. GOORMAGHTIGH, Bruges, Belgium

By a method similar to J. R. Musselman's (This Monthly, 1932, p. 290), it will be easy to prove the following theorem, generalizing several properties given in Musselman's paper and also a theorem proposed by W. H. Echols (This Monthly, vol. 39, 1932, p. 46):

Let $A_iB_iC_i$ be n equilateral coplanar triangles of the same rotation-sense ($i = 1, 2, \cdots, n$); at the vertices of each of the triangles $A_iB_iC_i$ are placed weights p_i; then the centers of gravity of the weights placed at the points A_i, at the points B_i and at the points C_i form an equilateral triangle.

When $n = 2$, we find that the points dividing A_1A_2, B_1B_2, C_1C_2 in the same ratio form an equilateral triangle.

When $n = 3$, the points having the same barycentric coordinates in the triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ form an equilateral triangle.

The first part of Professor Echol's theorem will be obtained when $n = 4$, $A_2 \equiv A_1$, $B_3 \equiv B_1$, $C_4 \equiv C_1$ and $p_2 = p_3 = p_4 = -p_1$; the center of gravity of the weights placed at A_1, A_2, A_3, A_4, is then the mid-point of A_2A_4. The second part is a particular case of the property found above for $n = 2$.

Concerning the Solution of Certain Types of Equations

By E. C. KENNEDY, Texas College of Mines

To solve equations of the type

$$(X + a)^n(X + b)^m(X + c)^p \cdots = K,$$

let X_0 be an estimated approximation to one of the real roots of the equation.

Set $X = X_0(1+u)$ and obtain

$$n \log\left[(a + X_0)\left(1 + \frac{X_0 u}{a + X_0}\right)\right] + m \log\left[(b+X_0)\left(1 + \frac{X_0 u}{b + X_0}\right)\right] \cdots = \log K$$

Since $\log(1+t) = t$, approximately, for small values of t, we obtain a much better approximation, X_1, from $X_1 = V/M + X_0$, where

$$V = \log K - n \log (a + X_0) - m \log (b + X_0) \cdots$$

$$M = \frac{n}{a + X_0} + \frac{m}{b + X_0} \cdots .$$

A five place table usually yields results correct to five places upon the first or second application. To obtain ten place accuracy a ten place table would probably be needed. However, even for great accuracy we do not need the log of a number of more than four or five digits—which we can read directly from any one of several well known tables. Also the method does not necessitate evaluating an antilog. Usually the quickest way to find X_0 (to about three places) is to interpolate using a log-log slide rule. Then one application of the formula should give the result correct to about six places.

Let us illustrate the advantages of this formula by

Example I. Solve

$$\frac{(X + 1)^{.511}(X + 3)^{1.4}}{(X + 2)^{.3}} = 20.4.$$

Here $n = .511$, $m = 1.4$, $p = -.3$, $K = 20.4$, $a = 1$, $b = 3$, $c = 2$. Taking $X_0 = 4$, we write at once

$$V = \log 20.4 - .511 \log 5 - 1.4 \log 7 + .3 \log 6 = .00637$$

$$M = .511/5 + 1.4/7 - .3/6 = .2522$$

and $X_1 = .00637/.2522 + 4 = 4.0253$, the first approximation. Using $X_1 = 4.025$ a second approximation, X_2, is readily found to be 4.02529231.

Example II. Solve $X^3 + AX^2 + BX = K_1$.

To solve this equation we put it in either one of the two forms

$$X(X + R_1)(X + R_2) = K_1, \text{ or}$$

$$X^2(X + A)(X - K_1/B)^{-1} = -B,$$

(where R_1 and R_2 are easily determined, being roots of a quadratic) and proceed as in I. The latter form is perhaps preferable. To illustrate let us solve

$$X^3 + 2X^2 + 3X = 91 \text{ or}$$

$$X^2(X + 2)(X - 91/3)^{-1} = -3.$$

Here $K = -3$, $n = 2$, $m = 1$, $p = -1$, $a = 0$, $b = 2$, $c = -91/3$. Taking $X_0 = 3.7$, we have

$$V = \log 79.9 - 2\log 3.7 - \log 5.7 = .02365,$$

$$M = 2/3.7 + 1/5.7 + 3/79.9 = .7534,$$

and

$$X_1 = .02365/.7534 + 3.7 = 3.7314 \text{ (Newton's method gives } 3.7317)$$

$$X_2 = 3.7314928.$$

The quintic $X^5+AX^4+BX^3+CX^2+DX=E$ may be solved by this scheme by putting it in the form

$$X^3(X+a)(X+b)(X+c)^{-1}(X+d)^{-1} = -C,$$

where $(X+a)(X+b)=X^2+AX+B$ and $-C(X+c)(X+d)=E-DX-CX^2$, provided these quadratics both have real zeros.

Example III. This method is especially useful on equations of the type $X^n+aX^{n-1}=K$. For example in solving $X^{11}+aX^{10}=K$ we set $X_1=X_0(1+u_1)$ and obtain

$$u_1\left(10+\frac{X_0}{a+X_0}\right) = \log\frac{K}{X^{10}(a+X_0)}.$$

In the equation $X^{11}+X^{10}=2.5$, $a=1$ and $X_0=1$, say. We write immediately $u_1(10+\frac{1}{2})=\log 2.5/2=.22314$ whence $u_1=.0213$ and $X_1=1.0213$ (Newton's method gives 1.0238). Taking $X_1=1.021$ we write, without any preliminary calculations,

$$u_2(10+1.021/2.021) = \log\frac{2.5}{(1.021)^{10}(2.021)},$$

which gives X_2 correct to about eight places. We may obtain X_3, correct to about fifteen places, from $u_3=(V+T)/M$; where V is the right side of the equation above, M is the coefficient of u_2, and $T=[5+\frac{1}{2}(1.021/2.021)^2]u_2^2$ is obtained from the second term of the expansion $\log(1+t)=t-t^2/2\cdots$. To find X to this degree of accuracy by Horner's Method or by interpolation would be very laborious.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

An Introduction to the Theory of Canonical Matrices. By H. W. Turnbull and A. C. Aitkin. London, Blackie and Sons, 1932. xiii+192 pages. 17 sh. 6d.

Integralgleichungen unter besonderer Berucksichtigung der Anwendungen. By G. Wiarda. Leipzig, B. G. Teubner, 1930. 184 pages. 8.64 marks.

Recent Developments in the Teaching of Geometry. By J. Shibli. State College, Pa., Published by the Author, 1932. x+252 pages. $2.25.

Introduction à la mécanique des fluides. By Adrien Foch. Paris, Librairie Armand Colin, 1932. vi+200 pages, 10 fr. 50.

Mathematische Grundlagen der Quantenmechanik. By J. von Neumann. Berlin, Julius Springer, 1932. 262 pages. Rm. 18.

Ergebnisse der Mathematik und ihrer Grenzgebiete, herausgegeben von der Schriftleitung des "Zentralblatt fur Mathematik." Erster Band. Berlin, Julius Springer, 1932.

1. *Knotentheorie*. By K. Reidemeister. vi+64 pages. Rm. 8.75.
2. *Graphische Kinematik und Kinetostatik*. By Karl Federhofer. vi+112 pages. Rm. 13.15.
3. *Lamêsche—Mathieusche—und verwandte Funktionen in Physik und Technik*. By M. J. O. Strutt. viii+116 pages. Rm. 13.60.
4. *Die Methoden zur angenäherten Lösung von Eigenwertproblemen in der Elasktokinetik*. By K. Hohenemser. 90 pages. Rm. 13.60.
5. *Fastperiodische Funktionen*. By Harald Bohr. 96 pages. Rm. 11.40.

Plane and Spherical Trigonometry. By G. N. Bauer and W. E. Brooke. Third Revised Edition. New York, D. C. Heath and Company, 1932. xvi+236 pages; tables, iv+140 pages. $2.00.

David Hilbert, Gesammelte Abhandlungen. Erster Band, Zahlentheorie. Berlin, Julius Springer, 1932. xiv+540 pages. Rm. 48.

Georg Cantor, Gesammelte Abhandlungen. Edited by Ernst Zermelo. Berlin, Julius Springer, 1932. viii+486 pages. Rm. 48.

Mathematics of Finance. By Rietz, Crathorne, and Rietz. Revised Edition. New York, Henry Holt and Company, 1932. xvi+346 pages. $3.00.

Introduction to Trigonometry and Analytic Geometry. By E. B. Skinner. New New York, The Macmillan Company, 1932. xii+190 pages. $1.80.

An Arithmetic for Teachers. By W. F. Roantree and Mary S. Taylor. New York, The Macmillan Company, 1932. x+524 pages. $2.50.

Cambridge Tracts in Mathematics and Mathematical Physics. General Editors G. H. Hardy and E. Cunningham. Cambridge University Press, 1932. New York, The Macmillan Company.

No. 27. *Modular Invariants*. By D. E. Rutherford. viii+84 pages. $2.00.

No. 28. *Conformal Representation*. By C. Caratheodory. viii+106 pages. $2.25.

No. 29. *The Foundations of Differential Geometry*. By Oswald Veblen and J. H. C. Whitehead. x+98 pages. $1.75.

Statistical Methods for Research Workers. By R. A. Fisher, Fourth Edition, Revised and Enlarged. Edinburgh and London, Oliver and Boyd, 1932. xvi +308 pages, with tables. 15 shillings.

The Theory of Functions. By E. C. Titchmarsh. Oxford, The Clarendon Press, 1932. x+454 pages. $7.50.

Elementary Mathematical Analysis, with Tables. By Mayme I. Logsdon. Volume 1. New York, McGraw-Hill Book Company, 1932. xiv+312 pages, tables 32 pages. $2.25.

REVIEWS

Vorlesungen über Grundlagen der Geometrie. By Kurt Reidemeister. Die Grundlehren der Mathematischen Wissenschaften, volume 32. Berlin, Julius Springer, 1930. x+147 pages. 12.60 marks.

This new work in Springer's well known series brings us a picture of some of the more recent contributions to the study of the foundations of geometry, particularly those made by a group of German mathematicians to which Reidemeister and Blaschke belong.

Reidemeister's work is restricted almost entirely to the plane, and his chief aim seems to be the setting up of an axiomatic basis for the study of affine and projective geometries. Two distinct approaches to this task are considered—the analytic and the axiomatic, and they motivate the division of the book into two parts. In the first part Reidemeister discusses the foundations of algebra, and bases affine and projective geometries on them. In the second part, a purely axiomatic development of geometry is based on the concept of a 3-web (3-Gewebe), and its completeness and consistency are demonstrated by forming from it a number system satisfying the postulates embodied in the first part.

The opening chapter on groups of transformations deals abstractly with such fundamentals as the "congruence" of elements under a group of transformations, the "reference set" (Bezugsmenge), a set of elements invariant only under the identical transformation (as, e.g., a set containing four linearly independent points under projective transformations), and the "natural coördinates" of an element with respect to a basic reference set.

In the second chapter number systems are developed on an axiomatic basis. The main purpose here is to lay the foundations for later geometric developments, but under the guise of independence proofs for his system of postulates, Reidemeister exhibits a number of algebras which fail to satisfy one or another of the usual axioms. These algebras are of considerable interest on their own account, particularly one unilaterally distributive number system attributed to Dickson.

The construction of affine and projective geometries on the number system as a basis occupies the next chapter. Vectors are subjected only to right-handed multiplication by a scalar factor, and only right-handed linear dependence is considered; commutativity of multiplication is thus cast aside in the development. Consequently determinants are not used, and so opportunity is afforded the author to exhibit considerable ingenuity in his algebraic developments.

In the second part, Reidemeister returns to the beginning of his problem and attacks it from the geometrical angle. The working basis is the 3-web, three pencils of parallel lines in a plane, and this concept is developed axiomatically from the undefined terms "point," "line," and "incidence." Both non-homogeneous and homogeneous coördinates are introduced; to each of the pencils is assigned an ideal "improper" point as vertex, and the ideal points of their tri-

angle are also termed "improper." A fourth pencil, with vertex at a proper point, completes the basic picture.

Addition and multiplication, proportionality and equality, brought in in terms of operations on vectors, permit the introduction of vector ratios in the role of numbers. The introduction of a number of closure axioms to the concept of a 3-web turns out to be equivalent to the introduction of distributive and commutative laws. In this way affine geometry is developed, together with the accompanying algebra. As in the earlier part, the lack of necessity for commutative multiplication is noteworthy.

The book closes with a chapter on the development of projective geometry, in which cross-ratio takes a prominent part. Here again non-commutative multiplication plays a role—witness the fundamental theorem of projective geometry, essentially in the following form: If v is the cross-ratio of four collinear points, and v^* the corresponding cross-ratio of the transformed points, then there exists a number v_0 such that $v^* = v_0 v v_0^{-1}$.

The author has obviously put a great deal of time and effort into the details of his work, and should be complimented on his discussions at the beginning and end of each chapter and of the book. One criticism which might be made is that the desire for logical compactness is at times carried to extremes. The necessity for the reader to keep in mind so many aspects at once sometimes makes the book rather hard reading. In this connection an index would have been of great help.

About a dozen errors were noted in reading through the book. A few of these were obviously typographical, but a number of them which were algebraic must have been present in the original copy. As in most of the works of this series, the printing is a distinct credit to the publisher.

Both as an original composition and for its collection of interesting developments, this work is a distinct addition to geometrical literature, and should appeal strongly to anyone deeply interested in the logical foundations of geometry.

ROBIN ROBINSON

Mathematics of Finance. By B. H. Crenshaw, Z. M. Pirenian, and T. M. Simpson. New York, Prentice-Hall, Inc., 1930. xiv+383 pages. $3.75.

This book is a text designed for students of commerce and business administration and should prove suitable for a one-year course given in these departments. The text is divided into two parts, the first part providing sufficient background in algebra for the student to undertake the study of the latter part, the mathematics of finance and life insurance.

The first part is devoted to a survey of elementary algebra and its application to commercial problems, being a treatment of linear equations, ratio, proportion, percentage, quadratic equations, the binomial theorem, construction of graphs, and logarithms. The second part offers a rather comprehensive treatment of compound interest, the annuity certain, amortization and sinking fund

methods, valuation of bonds, and includes the theory of depreciation and the mathematics of life insurance. There is a reasonable supply of exercises with illustrative examples except in the chapter on the mathematics of depreciation.

Only fifteen pages are devoted to the treatment of life insurance, which is sufficient for the purpose of the book. The authors deviate in only one place from the accepted notation of the actuary. We find the calculation of net single premiums preceding that of the contingent annuity and in the subsequent calculation of the annual premium no mention is accorded the fact that it may be regarded as an annuity.

J. H. BUSHEY

La Gamme. By P. J. Richard. Paris, Hermann et Cie., 1930. viii+231 pages.

In this book, the author has set down the physical and mathematical foundations of the various musical scales which have been in use in the western world from the days of Pythagoras to the present, excluding the whole-tone scale exploited by Debussy and others. The reason for the omission of this familiar seven-tone scale can only be inferred from M. Richard's remark about a scale of eight tones separated by equal intervals, which, he says, "would give an impression of painful monotony." Originally written for his daughter, the book is full of information, presented in a clear and readable style, for students of music or mathematics and indeed for "everybody who will not be frightened by the dryness of the first chapter."

This first chapter is an arithmetical introduction in which the properties of fractions are dealt with at length followed by a very summary treatment of exponents and logarithms. The work in fractions goes beyond the elementary operations only to the extent of proving two theorems: (1) By adding numerators and denominators in two unequal fractions, a third is formed intermediate in value between the two given; and (2) If two fractions, a/b and c/d, are such that $ad-bc=1$, every fraction, whose value is intermediate, has terms larger than those of the originals. It is interesting to note that musicians may claim priority in the use of logarithmic language, for long before Napier they spoke of *adding* two intervals to produce a third, knowing that, in fact, it is the *product* of the frequency-ratios characterizing the component intervals which gives the ratio for the resultant interval. An interval is thus the logarithm of the ratio characterizing it.

Chapters II to VI construct and discuss the scales of Pythagoras, Aristoxenus, Zarlino, Delezenne, that known as the "musicians scale," and the tempered scale of Rameau and Bach, then, after a necessary discussion of acoustics, the "natural scale," based on harmonics. Chapter VII is given to the old Greek and Church modes and the modern minor mode. So far the discussion has been on the basis of vibrating strings, but Chapter VIII takes up sounding pipes, Chapter IX, concords and dissonances resulting from tones sounding simultaneously and, in conclusion, Chapter X, a very illuminating discussion on the

relation between the sensations of listening to an orchestra and the nice ratios which express the tones of the scale mathematically.

The author has proposed to avoid "all considerations of a purely aesthetic nature" but this, of course, is neither desirable nor possible for one whose interest is primarily musical. There are characteristic notes on futurist noise-orchestras (p. 118), "the melancholy gaiety of jazz" (p. 168), and the new tone-qualities, "perhaps a gain," resulting from the use of mutes (p. 188).

GORDON H. GRAVES

Le Calcul des Différences Finies et ses Applications. By Alfred Henry. Translated by A. Sallin. Paris, Hermann, 1932. 210 pages. 50 francs.

This book is a valuable exposition of the elements of the Theory of Finite Differences as this theory relates to problems in interpolation, finite summation, and approximate integration. It also includes a brief introduction to Differential and Integral Calculus and a chapter on Probability.

The arrangement of the work is as follows (the A, B, C headings are the reviewer's).

(A) Chapters 1–9 deal with the development of the Difference Calculus and applications (pp. 11–96).

(B) Chapters 10–19 treat Differential and Integral Calculus (pp. 97–170).

(C) Chapter 20 is on Approximate Integration and Chapter 21 gives a brief treatment of some typical problems in the Theory of Probability (pp. 171–210).

We believe that the main purpose of the author has been to produce a text which would enable the actuary without much mathematical background to gain sufficient knowledge of the Calculus of Differences so that, aided by a superficial knowledge of differential and integral calculus, he could intelligently handle a wide class of problems in interpolation, finite summation, and approximate integration. We believe that the book accomplishes this purpose admirably. Also we find in Chapter 21 a treatment of Probability along with an interesting set of problems. Chapter 21 seems to have been added so that the book would have more appeal to the potential reader. It is in no way connected with the Theory of Finite Differences.

Chapters 1–4 introduce the simpler formulas of interpolation. The difference $\Delta f(x)$ is defined as $f(x+h)-f(x)$ where one is to determine h by the context. This notation avoids confusion in a textbook introducing the beginner to methods of interpolation. We believe, however, that in theoretical work the notation

$$\underset{h}{\Delta} f(x) = [f(x+h) - f(x)]/h$$

is more satisfactory. The operator $Ef(x) = f(x+h)$ is employed frequently in proofs involving operational methods. Chapter 5 takes up the central difference interpolation formulae of Stirling, Bessel, Gauss and Everett, and gives an exceptionally clear discussion of them. Chapter 6 is on Inverse Interpolation.

The reviewer finds Chapter 7 on Finite Summation novel and ingenious and thinks that this topic is worthy of more consideration in present day teaching than it now receives. The author in an exercise presents the following method of summing a convergent alternating series

$$\phi(1) - \phi(2) + \phi(3) - \phi(4) + \cdots$$

The sum is given by

$$\tfrac{1}{2}\phi(1) - \tfrac{1}{4}\Delta\phi(1) + \tfrac{1}{8}\Delta^2\phi(1) - \cdots$$

Thus to evaluate the series

$$\frac{1}{10} - \frac{1}{11} + \frac{1}{12} - \cdots$$

to four decimal places, four terms of the difference series will suffice as against an odd 10,000 of the original series. There are several misprints in this chapter. In the table on page 76, the 2nd column should be headed Intégrales indéfinies, and in the 3rd and 4th rows of the table, the superscripts should be inclosed in parentheses or brackets to show that the functions represent factorials (defined on page 31). The $f(a)$ under the summation sign, line 11, p. 77, should read $f(x)$. The superscripts 1, 2, 3 in the 2nd line of Ex. 4, p. 80, should be subscripts.

Chapter 8 gives a brief discussion of Divided Differences[1] and Chapter 9 an intelligent treatment of Differences of Functions of Two Variables relative to interpolation.

Part (B) on the Integral and Differential Calculus is a brief exposition of basic technical results without much attempt to go below the surface in the theory. The reviewer believes that such a treatment might well be accompanied by frequent references which would put the beginner on his guard against the many pitfalls that one may encounter in using limiting processes. With the exception of Chapter 15, Part (B) is entirely unconnected with the Theory of Finite Differences. In Chapter 15, however, it is shown that the derivative and the difference of a function are related by the symbolical equation

$$\Delta = e^{hD} - 1,$$

where D is the operational symbol for differentiation. This relation is effectively used to obtain the Euler-Maclaurin summation formula in Chapter 20, although the justification of this use is not discussed. Chapter 20, on Approximate Integration, takes up, besides the Euler-Maclaurin formula, formulae of Woolhouse, Lubbock, Simpson, Weddle, G. H. Hardy, Lagrange, the Rule of Three Eighths and one other, not bearing a name.

As stated above, the last chapter, Chapter 21, is on Probability. Except for the errors in Chapter 7 noted above, there are few typographical errors.

B. F. KIMBALL

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[1] cf. Whittaker & Robinson, *A Short Course in Interpolation*, London, Blackie & Son (1923), Chapter 2.

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

A.

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research in the preparation of papers in the field of mathematical science to be presented at its regular meetings.

CHAPTER REPORTS

1931–1932

Pi Mu Epsilon of Hunter College.

The Hunter College chapter of Pi Mu Epsilon has just finished another busy year, consisting of eight program meetings, two business meetings, and three social functions. At the program meetings during the first semester the topic discussed was "Maxima and minima." Eleven girls gave papers on various phases of the subject. The second semester topic was "Algebraic and transcendental numbers," and eleven girls reported on this topic also.

The chapter initiated twenty new members, making the membership for the year, including the faculty members, total fifty-three.

The October social function was a dinner for initiates. Many alumnae members attended. Interesting talks were given by former active members who told of their work in different fields. The February party was a luncheon. Professor E. R. Hedrick of the University of California at Los Angeles was our guest and speaker. The May party was an alumnae reunion and bridge party.

The officers for the year 1931–1932 were: Professor Jewell C. Hughes, Director; Sylvia Radlich, Vice Director; Rose Berger, Corresponding Secretary; Madeline Levin, Treasurer; Isabel Sklower, Recording Secretary.

ROSE BERGER, *Corresponding Secretary*

Pi Mu Epsilon of the University of California at Los Angeles.

We extend our best wishes to the other chapters of Pi Mu Epsilon. Our chapter presents the following account of its activities for the academic year 1931–1932:

Our officers were: Mr. Carroll P. Brady, Director; Miss Goldie Ivener, Vice Director; Mr. Hugh J. Hamilton, Secretary; Mr. Wendell E. Mason, Treasurer; Professor Raymond Garver, Librarian; Professor Clifford Bell, Faculty Advisor.

Twelve new members were elected on December 5, 1931, and three on April 30, 1932, so that the total membership in this chapter is now 117. Thirty-five of these are now at the University, thirteen being faculty members.

Since September there have been seven regular meetings and two business sessions. Two entertainments were given in honor of incoming members, an informal one on December 5, 1931, and a semi-formal one on April 30, 1932. Our chapter has also sponsored, in conjunction with the Mathematics Club of this University, a Christmas party on December 18, 1931, and a Beach party on June 14, 1932.

In order to stimulate interest in the Calculus, this chapter directs annually a Prize Calculus Examination, open to all students save members of Pi Mu Epsilon. The winner is awarded ten dollars.

The meetings and programs were as follows:

October 7, 1931: "Skew curves" by Mr. Hugh Hamilton.

November 18, 1931: "Some concepts of the calculus" by Professor G. E. F. Sherwood.

January 13, 1932: "An application of Euler's method for the summation of a certain class of series" by Mr. Carroll Brady.

March 9, 1932: "Business mathematics" by Miss Goldie Ivener.

April 6, 1932: "The problem of Apolonius" by Miss Jean Robb.

May 4, 1932: "The mathematics of rainbows" by Mr. R. Holloway.

June 1, 1932: "Researches on sound waves" by Mr. Reed Lawlor.

HUGH J. HAMILTON, *Secretary*

B.

LOCAL MATHEMATICS CLUBS

The Junior Mathematical Club of the University of Chicago

The officers, elected by vote of the club on June 10, 1931, for 1931–1932 were: Mr. Magnus R. Hestenes, President; Mr. Ralph D. James, Secretary-Treasurer; Miss Julia W. Bower, Social Chairman; Miss Frances H. Wiancko, Program Chairman.

The primary aim of the Junior Mathematical Club is to broaden the student's knowledge of mathematics and to supplement class room instruction by giving students an opportunity to present and to listen to papers on various phases of mathematics. The Club largely sponsors the social activities of the Department of Mathematics. Its activities are quite distinct and in addition to the work of the graduate research club. The membership is about seventy-five, any graduate or undergraduate student majoring in mathematics and mathematical astronomy being eligible to membership. Meetings are held fortnightly on Wednesday, a social half-hour preceding the presentation of the papers.

The meetings and programs were as follows:

October 7, 1931: Address of welcome to new students by Professor E. P. Lane; "The use of the library" by Professor L. M. Graves.

October 21, 1931: "Early developments of the calculus" by Dr. R. G. Sanger.

November 4, 1931: "Non-differentiable functions" by Dr. E. J. McShane.

November 18, 1931: "On the problem of moments for a finite interval" by Dr. I. Schoenberg.

December 2, 1931: "Non-Euclidean geometry" by Professor M. I. Logsdon.

December 16, 1931: "Elementary properties of ordinary linear differential equations of the second order" by Dr. W. T. Reid.

January 20, 1932: "Historical sketch of synthetic projective geometry" by Mr. P. Youtz.

February 3, 1932: "Some properties of orthogonal matrices and an application to analytic geometry of space" by Mr. R. C. Bullock.

February 17, 1932: "Classification of elements of a group" by Professor A. C. Lunn.

March 2, 1932: "Diophantus of Alexandria" by Mr. Ralph Hull.

March 30, 1932: "Mathematical models" by Professor E. P. Lane.

April 13, 1932: "Some problems in symmetric functions" by Miss Julia Bower.

April 27, 1932: "On minimum problems in geometry" by Dr. I. Schoenberg.

May 11, 1932: "Mathematics in the large" by Professor H. E. Slaught.

During this year, several bridge parties have been sponsored by the club.

RALPH D. JAMES, *Secretary-Treasurer*

The Mathematics Club of Hunter College.

During the fall term of the past year the program meetings of the Mathematics Club of Hunter College were devoted to the topic of Cartography. Several students reported on different types of

map construction, for example, Mercator's, Mollweide's, and Stereographic Projection. At one meeting, Mr. Nelson of the Rand McNally Company gave a summary of the types of map construction in actual use; and at the final meeting of the term, Professor Lehnerts, Head of the department of Geology at Hunter College, gave a talk on topographical maps. During the second term of the year, miscellaneous topics of mathematical interest were discussed. They were: (1) Geometrical constructions with a straight edge only; (2) The golden section; (3) The nine-point circle; (4) Spirals; (5) The conchoid of Nicomedes; and (6) Three Italian texts of the sixteenth and seventeenth centuries—translated by three students in the course in History of Mathematics.

In addition to the program meetings, several social affairs were held during the year. Each term, a party was given in honor of the new members of the Club. In the Fall, the Club at the Main Building was the guest of the Club at the New Buildings in the Bronx, and some two hundred were present at the party held there. At a Friday afternoon tea, Professor Walker told of some of her experiences on her trip around the world. Also, last Fall, there was a twelve mile Club hike attended by about fifteen students and two members of the Staff. The last party of the Club for the year was a Bridge party, held on the evening of April 30, 1932.

The officers for the year were: Irene Larson, President; Kathleen Downing, Vice President; Beatrice Jacobson, Secretary; Marian Moynahan, Treasurer; Marion Leary—Fall term, Helen Schroeder—Spring term, Publicity Manager; Laura Guggenbühl, Faculty Advisor.

BEATRICE JACOBSON, *Secretary*

The Mathematics Club of Boston University.

The Mathematics Club of Boston University, College of Liberal Arts was organized in 1922 to promote good fellowship among those interested in this science. This organization also affords the students and Professors an opportunity to discuss many interesting aspects of mathematics not covered in the regular courses offered by the college.

Under President Thomas Homkowycz several socials have been held during the past year. Early in the Fall an acquaintance party welcomed new members. A Christmas social and a theatre party in February constituted the winter program. The social activities were completed by an April Fool's party. But the serious side of a mathematics club has not been neglected. Papers have been given on the following subjects during the past year:

"Problems I have met" by Professor Elmer B. Mode; "Oriental numbers" by Miss Annie Hall; "The mathematics of the musical scale" by Mr. Lucien B. Taylor, Instructor in Physics; "Totients" by Miss Elizabeth E. Curtis; "The circular points at infinity" by Miss Ruth Irene Deffley; "Einstein" by Mr. Julius Miller; "The possible impossible" by Professor Robert E. Bruce, Faculty Advisor of the Club; and "Solutions of equations" by Mr. Edward Mann, Instructor of Mathematics at the College of the Eastern Nazarene.

The annual prize for the best paper was awarded to Mr. Miller.

The 1931-1932 Corps of Officers included: Mr. Thomas Homkowycz, President; Miss Gladys Knowlton, Vice President; Mr. Carleton Foss, Secretary; Miss Doris Atkinson, Assistant Secretary; Mr. Francis Blackwell, Treasurer; Miss Dorothy Brennan, Assistant Treasurer; Miss Eleanor Johnson and Miss Annie Hall, Executive Committee; Professor Robert E. Bruce, Faculty Advisor.

RUTH IRENE DEFFLEY, *Reporter*

The Dartmouth Mathematical Society.

The Dartmouth Mathematical Society was founded on October 6, 1930, "to make possible the presentation and discussion of subjects of general mathematical interest." Although primarily an undergraduate society, membership is open to anyone interested, upon payment of the regular dues of $1.00 per semester. There are at present thirteen members and frequently non-members are guests at the meetings.

Meetings are ordinarily held every second week during the college year. It is customary to serve refreshments during the discussion following the meeting.

The officers for 1931–1932 were: W. C. Johnson '33, President; C. Calmon '34, Vice President; and F. L. Engel '34, Secretary-treasurer.

The meetings and programs for the year were as follows:
October 5, 1931: "Fourth dimensional space" by Professor R. D. Beetle.
October 19, 1931: "The trisection of the angle" by F. L. Engel '34 and G. L. Engel '34.
November 2, 1931: "Mathematical concepts of infinity" by Dr. F. W. Perkins.
November 23, 1931: "The composition of simple harmonic motions" by G. F. Hull, Jr., '32.
December 14, 1931: "DeMontmort's problem" by Professor B. H. Brown.
February 15, 1932: "The development of the fundamental algebraic notations" by A. P. Bill '33.
February 29, 1932: "How mathematicians make generalizations" by Professor L. L. Silverman.
March 14, 1932: "The mathematics of ship building" by Dr. R. Robinson.
April 18, 1932: "Hyperbolic functions and the catenary" by W. C. Johnson '33.

FRANK L. ENGEL, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSEN AND W. F. CHENEY, JR.

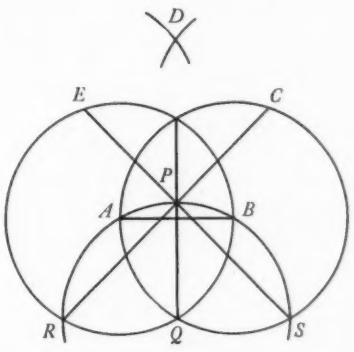
ELEMENTARY PROBLEMS

Send all communications about Elementary Problems to Wm. Fitch Cheney, Jr., Dept. Box 35 Storrs, Conn.

The Department of Elementary Problems and Solutions in the Monthly welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 6. *Proposed by W. R. Ransom, Tufts College.*



This construction was given in 1625 by Albrecht Dürer, the great engraver, for a regular pentagon, $ABCDE$, and it is still given in books on mechanical drawing. The circles are all drawn with the same radius, equal to the given length of the side AB, with centers at these points (in order) A, B, Q, C and E. Calculate the angle ABC to determine whether this is an exact or an approximate construction.

E 7. *Proposed by C. O. Oakley, Brown University.*

The following letters represent the digits of a problem in simple addition.

$$\begin{array}{cccc} & S & E & N & D \\ & M & O & R & E \\ \hline M & O & N & E & Y \end{array}$$

Solve and show that the solution is unique.

E 8. *Proposed by Otto A. Spies, St. Paul, Minn.*

It is required to construct an inscriptible quadrilateral with ruler and compass, given the lengths of the four sides in order.

E 9. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

The cylindrical tank on a railway tank-car is of length l feet and radius r feet. The car stands on a grade inclined Q degrees from the horizontal. The tank is being emptied, and it is desired to express the volume of liquid in the tank as a function of the vertical depth of liquid below the test-hole in the center of the top of the car.

This problem becomes more difficult if the tank has hemispherical ends.

E 10. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

The x's below show the relative positions of the digits in a certain exact division. Find the divisor.

$$\begin{array}{l} x\,x\,x\,)\,x\,x\,x\,x\,x\,x\,x\,x\,(\,x\,x\,x\,x\,x\,x \\ \quad x\,x\,x \\ \quad\overline{} \\ \quad x\,x\,x\,x \\ \quad\ \ x\,x\,x \\ \quad\overline{} \\ \quad\ \ \ x\,x\,x\,x \\ \quad\ \ \ x\,x\,x \\ \quad\ \ \overline{} \\ \quad\ \ \ x\,x\,x\,x \\ \quad\ \ \ x\,x\,x\,x \\ \quad\ \ \overline{} \end{array}$$

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3572. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

The sum of the three bimedians of a tetrahedron (i.e., the lines joining the mid-points of the pairs of opposite edges) is less than one half and greater than one fourth of the sum of the edges of the tetrahedron.

3573. *Proposed by Samuel I. Jones, Nashville, Tenn.*

A hawk, eagle and sparrow are in the air. The eagle is 50 feet above the sparrow and the hawk is 100 feet below the sparrow. The sparrow flies straight forward in a horizontal line. Both hawk and eagle fly directly towards the sparrow. The hawk flies twice as fast as the sparrow. The hawk and eagle reach the sparrow at the same time. How far does each fly and at what rate does the eagle fly?

3574. *Proposed by Orrin Frink, Jr., Pennsylvania State College.*

A company wishes to establish five agencies at five points of a circular region. When this is done there will be a smallest number r such that every point of the circular region will be within a distance r of at least one of the agencies. How should the agencies be located so as to make r a minimum?

3575. *Proposed by Frank Morley, Johns Hopkins University.*

Given two circles in a Euclidean space, which are not interlaced and not cospherical; show that there are four circles which touch both, and that these break into two pairs, the three pairs forming a symmetrical configuration.

3576. *Proposed by J. J. L. Hinrichsen, Iowa State College.*

Given any triangle with three line segments concurrent in a point interior to the triangle and joining each vertex to its opposite side; prove that the length of the longest of these three lines cannot be less than $\sqrt{3}/2$ times the length of the opposite side of the triangle.

3577. *Proposed by B. F. Kimball, Schenectady, N. Y.*

Given

$$v_n = \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^r + x_2^r + \cdots + x_n^r}{(x_1 + x_2 + \cdots x_n)^q} dx_1 dx_2 \cdots dx_n,$$

where q is any real number and r is any real number greater than -1; show that

$$\lim_{n\to\infty} n^{q-1} v_n = 2^q (r+1)^{-1}.$$

Note. This is a generalization of problem 3408 [1930, 38] and problem 3460 [1930, 508].

3578. *Proposed by J. M. Feld, Brooklyn College.*

Two points are isogonal conjugates with respect to a triangle if and only if they are the foci of an inscribed conic.

SOLUTIONS

3508 [1931, 409]. *Proposed by the late Artemas Martin, Washington, D. C.*

The sides of a plane triangle are a, b, c. It is required to determine the radius of the circle circumscribing the escribed circles of this triangle. (See the Annals of Mathematics, March, 1894.)

Solution by R. Goormaghtigh, Bruges, Belgium

Let $A\,B\,C$ be the triangle, A', B', C' the mid-points of the sides, R and r the radii of the circum- and incircles, I, I_1, I_2, I_3, the in- and ex-centers of ABC, I' the in-center of $A'B'C'$, α the projection of A' on I_2I_3, $2s$ the sum of the sides of ABC.

The radical axis of the circles (I_2), (I_3) passes through A' and is perpendicular to I_2I_3; therefore I' is the radical center of (I_1), (I_2), (I_3). The power of A' with respect to (I_2), (I_3) being $(b+c)^2/4$, the power π of I' with respect to (I_1), (I_2), (I_3) is

$$(1)\qquad \pi = (b+c)^2/4 - \overline{A'\alpha^2} + (A'\alpha - \tfrac{1}{2}AI)^2,$$
$$= \tfrac{1}{4}(b+c)^2 - \tfrac{1}{2}(s-a)(b+c) + \tfrac{1}{4}(s-a)^2 + \tfrac{1}{4}r^2 = \tfrac{1}{4}(s^2+r^2).$$

There are *eight* circles tangent to (I_1), (I_2), (I_3): the three sides, the nine point circle (O_9) and the inverses of these four circles, for the pole I' and the power π.

The power ω of I' with respect to (O_9) being $-\frac{1}{2}Rr$, the radius of the inverse of (O_9), which is the circle circumscribed to (I_1), (I_2), (I_3), will be

$$(2)\qquad R\pi/2\omega = \tfrac{1}{4}(r + s^2/r)$$
$$= \tfrac{1}{4}\left[s^{-1/2}(s-a)^{1/2}(s-b)^{1/2}(s-c)^{1/2} + s^{5/2}(s-a)^{-1/2}(s-b)^{-1/2}(s-c)^{-1/2}\right].$$

It may be noted that the radii of the three other circles, inverse to the three sides, and also tangent to the escribed circles, are

$$(3)\qquad \tfrac{1}{4}a(r + s^2/r)(b+c)^{-1}, \cdots.$$

A Note by Otto Dunkel. The following amplification may be of aid in reading this interesting solution. Let $BC=a$ be tangent to the escribed circles (I_2) and (I_3) at A_2 and A_3. Then, since BA_2 and CA_3 are each equal to $s-a$, A' is also the mid-point of A_2A_3. Thus $A'A_2=\frac{1}{2}[2(s-a)+a]=\frac{1}{2}(b+c)$. Since I_2I_3 is the external bisector of angle A, it is perpendicular to the internal bisector AI. The triangle $A'B'C'$ has its sides parallel to the corresponding sides of ABC, and hence the internal bisector $A'I'$ of angle A' is parallel to AI and perpendicular to I_2I_3. It then follows that $A'\alpha$ passes through I', and that I' is the radical

center of (I_1), (I_2), (I_3). The perpendiculars from B and C to I_2I_3 have the respective lengths $c\cos\frac{1}{2}A$, $b\cos\frac{1}{2}A$, and hence $2A'\alpha=(b+c)\cos\frac{1}{2}A$. Also $AI\cos\frac{1}{2}A=s-a$, and, therefore,

$$2A'\alpha\cdot AI=(b+c)(s-a).$$

The circle orthogonal to the three escribed circles has I' for center and the length $\pi^{\frac{1}{2}}$ for radius. If r_2 is the radius of (I_2) we see from a figure the following equations

$$\begin{aligned}\pi+r_2^2&=(\alpha I_2)^2+(\alpha I')^2,\\(A'A_2)^2+r_2^2&=(\alpha I_2)^2+(\alpha A')^2,\\\pi&=(A'A_2)^2-(\alpha A')^2+(\alpha I')^2\\&=(A'A_2)^2-(\alpha A')^2+(\alpha A'-I'A')^2.\end{aligned}$$

From the similarity of ABC and $A'B'C'$ we have $AI=2A'I'$. Also $AI^2=r^2+(s-a)^2$. These results give the expression for π in (1).

The nine point circle (O_9) of ABC with center O_9 is the circumcircle of $A'B'C'$ and hence its radius is $\frac{1}{2}R$. Let d be the distance between O_9 and I', the centers of the circum- and in-circles of $A'B'C'$. This is one half of the corresponding distance for ABC, and hence a well known theorem gives for d

$$4d^2=R^2-2rR.$$

The power ω of I' with respect to (O_9) is negative since I' lies within (O_9). It may be represented geometrically by constructing the chord TT' through I' perpendicular to O_9I'. Then

$$\begin{aligned}\omega&=-(I'T)^2=(I'O_9)^2-(O_9T)^2\\&=\tfrac{1}{4}(R^2-2rR)-\tfrac{1}{4}R^2=-\tfrac{1}{2}rR.\end{aligned}$$

Let the inverse of (O_9) with respect to the orthogonal circle above with radius $\pi^{\frac{1}{2}}$ be (O_9') with the center O_9'; let T and T' invert into $\overline{T}$ and $\overline{T}'$; let K be the pole of TT' with respect to (O_9); and let KI' cut (O_9) in M and N. Since K, M, I', N is an harmonic set of points, K inverts into the center of (O_9'). We then have in turn

$$I'K\cdot I'O_9'=\pi=I'T\cdot I'\overline{T};\ (I'T)^2=O_9I'\cdot I'K;$$

$$\frac{I'O_9'}{I'O_9}=\frac{I'\overline{T}}{I'T}.$$

Hence $I'O_9T$ and $I'O_9'\overline{T}$ are similar triangles, and it then follows that

$$O_9'\overline{T}=O_9T\frac{I'\overline{T}}{I'T}=\frac{1}{2}\frac{R\pi}{(I'T)^2}=-\frac{1}{2}\frac{R\pi}{\omega}=\frac{\pi}{r}.$$

This gives the result in (2).

If h_a is the altitude of ABC from A, the distance between BC and $B'C'$ is $\frac{1}{2}h_a$. Let p_a be the distance of I' from BC; then $p_a=\frac{1}{2}h_a-\frac{1}{2}r$, since the distance of I' from $B'C'$ is one half that of I from BC. Also $h_a=2rsa^{-1}$ as we find by obtaining the area of ABC in two ways. Hence

$$p_a = \frac{r}{2a}(2s - a) = \frac{r(b + c)}{2a}.$$

The side BC inverts into a circle with the diameter πp_a^{-1}, and hence the radius of this circle is

$$\frac{a\pi}{r(b + c)}.$$

3512 [1931, 461]. *Proposed by J. Rosenbaum, Milford, Conn.*

Prove that in the tetrahedron of Problem 3482 (March, 1931) the common center of the two spheres is also the centroid of the tetrahedron.

Solution by the Proposer

Consider the tetrahedron $ABCD$ with the faces ABD, BCD, and CAD opened up to the positions ABD_1, BCD_2, and CAD_3 forming the plane figure $D_1BD_2CD_3A$. Since the sum of the face angles at A, B, and C is each two right angles, the figure $D_1BD_2CD_3A$ is a triangle. In this triangle, A, B, and C are the midpoints of the sides. Hence the faces of $ABCD$ are congruent, and thus are equivalent.

It will now be proved that in a tetrahedron with equivalent faces the centroid and the incenter coincide; Join the incenter to the vertices thus dividing the tetrahedron into four tetrahedra. The entire volume is the sum of the four volumes, or

$$Ah/3 = 4Ar/3, \tag{1}$$

where A is the area of each face, h is the altitude to a face and r is the inradius.

From (1) we have

$$r = h/4, \tag{2}$$

from which the theorem above readily follows, and the statement of the problem is thus proved.

3515 [1931, 462]. *Proposed by E. P. Bugdanoff, Harbin City, China.*

Give an elementary proof, without the use of the calculus, that the equation

$$2^x = 4 \cdot x$$

has two real roots only, and calculate each of them.

Determine the location of the imaginary roots, and compute the pair having the least absolute value.

Solution by Otto Dunkel, Washington University

Setting $x=0, 1, 4$, we find that one root is 4 and that a second root lies between 0 and 1. In order to prove that these are the only two real roots, it suffices to show that a straight line cuts the curve (C), $y=2^x$, in not more than two real points. It can be proved by elementary algebra that

$$(1) \qquad \frac{a^\alpha - 1}{\alpha} > \frac{a^\beta - 1}{\beta}, \quad a > 0,\ a \neq 1,\ \alpha > \beta > 0,$$

where α and β are any two real numbers. For each value of x there is only one point on (C), and y increases continuously with x. Let a straight line cut (C) at A and B with abscissas x_1 and x_2, $x_2 > x_1$, and let P, with abscissa x, lie on (C) between A and B, then

$$(2) \qquad \text{slope } AB = 2^{x_1}\frac{2^d - 1}{d}, \quad \text{slope } AP = 2^{x_1}\frac{2^e - 1}{e},$$

$$x_2 - x_1 = d, \quad x - x_1 = e.$$

It follows from (1) that slope $AB >$ slope AP, and hence P lies below the secant AB. If P lies beyond B then the previous inequality is reversed, and P lies above the secant AB. If P lies to the left of A, we show in the same way that B lies above secant PA, and hence P lies above secant AB. This completes the proof that there cannot be more than two real roots, and the proof shows that (C) is concave upward. A simple method of calculating the remaining root r now follows from the rising and concave form of (C). Set

$$(3) \qquad y = 2^{x-2}.$$

If $r < x < 4$, say $x=1$, then $y > r$ and $y < x$. Hence y is a closer approximation to r than x. Using this value of y for x, we find a still better approximation y_1; setting $x = y_1$, we find y_2; etc. The root r may be approximated in this way to any degree of accuracy; but this method is not as rapid in the advanced stages of approximation as the so-called Newton's method. To four decimals $r = 0.3099$.

The formula (1) is important, since from it may be deduced the derivatives of all powers of x, of the exponential functions, and of the logarithmic functions. The proof of this formula by algebra is not simple when compared with the proofs found in most of our texts for undergraduates.

In order to determine the imaginary roots we set $x = \rho e^{i\theta}$, $\theta \neq 0$, and find the equations

$$(4) \qquad \theta \cot\theta - \log_e\left(\frac{\theta}{\sin\theta}\right) - c = 0, \quad c = \log_e\left(\frac{4}{\log_e 2}\right) = 1.75280,$$

$$(5) \qquad \rho = \frac{\theta}{\log_e 2 \sin\theta}.$$

It will suffice to consider $\theta > 0$; then from the above equations $\sin\theta > 0$. Since $\theta/\sin\theta > 1$, $\theta\cot\theta > c > 1$, or $0 < \tan\theta < \theta$; and this shows that no root of (4) lies

between 0 and $\pi/2$. Roots can occur only in the intervals $2k\pi \leqq \theta \leqq (4k+1)\pi/2$, $k=1, 2, \cdots$. If we set y equal to the left side of (4), then

$$\frac{dy}{d\theta} = -\theta^{-1}[(\theta \cot\theta - 1)^2 + \theta^2], \tag{6}$$

and therefore y decreases as θ increases. For $\theta = 2k\pi+$, $y = +\infty$; for $\theta = (4k+1)\pi/2$, $y<0$. This shows that there is one and only one root in each of the intervals. If θ_1 is a root in a given interval, then for $\theta = 2\pi+\theta_1$ in the next interval

$$y = (2\pi + \theta_1)\cot\theta_1 - \log_e\left(\frac{2\pi+\theta_1}{\sin\theta_1}\right) - c,$$

$$0 = \theta_1 \cot\theta_1 - \log_e\left(\frac{\theta_1}{\sin\theta_1}\right) - c, \tag{7}$$

$$y = 2\pi\cot\theta_1\left[1 - \frac{\tan\theta_1}{2\pi}\log\left(1+\frac{2\pi}{\theta_1}\right)\right] > 2\pi\cot\theta_1\left[1 - \frac{\tan\theta_1}{\theta_1}\right]$$

$$> 2\pi\cot\theta_1[1 - c^{-1}] > 0.$$

Hence the root in the next interval lies nearer the right hand end of that interval than the root in the preceeding interval. It will appear later that the roots approach as a limit the right hand end of the intervals. It will be found by trial for the first interval that the root lies beyond $2\pi+\pi/3$, and hence for any root

$$2k\pi + \pi/3 < \theta < 2k\pi + \pi/2,\ k = 1, 2, \cdots. \tag{8}$$

In order to examine the variation of ρ we consider the equation

$$\log_e 2\cos\theta = \rho^{-1}\log_e(4\rho). \tag{9}$$

Since from (5) $\rho>1$, we see that the right side of (9) decreases as ρ increases. On the left side $\cos\theta$ decreases as we pass from the root in one interval to the root in the next, as has been shown above. Hence ρ increases as we pass from the root in one interval to that in the following interval.

In the order to calculate the roots in a simple manner, set $\theta = M\pi - \alpha$, $M = (4k+1)/2$; then

$$\tan\alpha = \frac{\log_e(M\pi-\alpha) - \log_e\cos\alpha + c}{M\pi - \alpha},\ 0 < \alpha < \pi/6. \tag{10}$$

From this expression we see that α approaches zero as M increases without limit. From the form of (10) we see that α may be approximated as follows: replace $\tan\alpha$ on the left by $\tan\alpha'$; for the first interval select any suitable trial value for α, say $\alpha=\pi/6$, and find α', $\alpha' = 27°46.5'$ approximately. Continue, using $\alpha = 27°46.5'$, and after a few steps we get $\alpha' = \alpha = 27°43'$ approximately. The convergence becomes, of course, more rapid as we proceed with the intervals. The results for the first five intervals for (ρ, α) are

(12.01, 0.4838); (20.95, 0.3099); (29.93, 0.2327);

(38.94, 0.1881); (47.96, 0.1601).

The root in the first interval is $10.63+5.59i$. To each such root there is a corresponding conjugate root.

Calculus has been used in (6) to save space; but it may be shown by trigonometry that $\theta\cot\theta$ decreases and $\theta/\sin\theta$ increases as θ increases in any of the given intervals. This gives the result used. Also the right side of (9) may be shown to decrease under the given condition by use of algebra; but the proof is rather longer than by the use of calculus.

Also solved by F. L. Wilmer.

3519[1931, 589]. *Proposed by Norman Anning, University of Michigan.*

AB is a fixed diameter and CD is a moving diameter of a given circle $ACBD$. E is the mid-point of arc BC. DE and AC intersect in P. Prove that DP is normal to the locus of P.

Solution by Mrs. Ruth B. Smith, Oberlin, Ohio

Choose the fixed circle $ABCD$ so that its center is at $(a, 0)$ and its radius is a. The coordinates of C are easily found to be $(2a\cos^2\alpha, a\sin 2\alpha)$, where $\alpha=\angle BAC$. The coordinates of D, $(2a\sin^2\alpha, -a\sin 2\alpha)$, easily follow from those of C. The coordinates of E are $[a(1+\cos\alpha), a\sin\alpha]$, and the equation of DE is then found to be

$$(1)\qquad (\sin 2\alpha+\sin\alpha)x-(\cos 2\alpha+\cos\alpha)y-a(2\sin\alpha+\sin 2\alpha)=0.$$

The coordinates of P are obtained by solving (1) with $y=x\tan\alpha$, and they are found to be

$$(2)\qquad x=2a\cos\alpha(1+\cos\alpha),\quad y=2a\sin\alpha(1+\cos\alpha).$$

The slope of the locus of P is

$$(3)\qquad \frac{dy}{dx}=-\frac{\cos\alpha+\cos 2\alpha}{\sin\alpha+\sin 2\alpha},$$

and, since it is the negative reciprocal of the slope of (1), the line DE is normal to the curve.

In Cartesian form the equation of the locus of P is

$$x^2+y^2-2ax=2a(x^2+y^2)^{1/2},$$

which is the equation of a cardioid with its cusp at the origin.

A Note by Otto Dunkel. If O is the center of the circle $ABCD$, OE is parallel to CP, and therefore $CP=2OE=AB$. Hence the locus of P is a cardioid. Since CP has a constant length, its instantaneous center of rotation I is the intersection of the normal to the circle CD and the normal PI to the locus of P. Since CP passes through the fixed point A, the perpendicular to PA at A passes through I. Hence $I\equiv D$, and DP is normal to the locus of P.

Also solved by R. P. Agnew, S. F. Bibb, A. D. Bradley, J. H. Butchart, C. S. Carlson, Mannis Charosh, Rufus Crane, N. C. Fisk, Edward Fleisher, R. Goormaghtigh, H. Grossman, L. S. Johnston, Roy MacKay, V. F. Murray, A. Pelletier, W. H. Rasche, H. D. Ruderman, William Sell, L. S. Shively, E. Siroky, Wallace Smith, F. Underwood, J. M. West, and Roscoe Woods.

3520 [1931, 589]. *Proposed by Elijah Swift, University of Vermont.*

To construct a triangle, given the base BC, the opposite angle, A, and the length of the bisector, t, of the angle A. Show the construction is possible with ruler and compass and give a simple construction.

Solution by Richard Morris, Rutgers University

On BC as chord construct a circle containing angle A in one of its segments. The mid-point, K, of the arc of the segment containing the supplement of angle A is a fixed point, and chord, BK, becomes known, since the length of bisector t is given. A perpendicular from K upon BC locates D on BC and E on the circle. The line AK would intersect BC at U. The right triangles BEK and BDK are similar, hence $(BK)^2 = KD \cdot KE$. Also triangles KAE and KDU are similar, and hence $KD \cdot KE = KA \cdot KU = (BK)^2$, or $KA(KA - t) = (BK)^2$. Thus

$$KA = \tfrac{1}{2}\left[t \pm \left\{t^2 + (2BK)^2\right\}^{1/2}\right].$$

Now construct a right triangle whose legs are t and $2BK$. The hypotenuse of this triangle plus t equals twice KA. With K as center and KA as radius the vertex A is located on the circumcircle. There will be 2, 1, or 0 solutions. Also the hypotenuse minus t equals twice KU. The radius KU locates U on BC and the line KU produced locates A on the circle. As before there will be 2, 1, or 0 solutions.

The same type of construction may be used to obtain the triangle if the external bisector is given.

The solution presented is a slight modification of that suggested in Casey's Sequel to Euclid, edition of 1886 page 80.

A Note by the Editors. There is no real construction if $2t \tan\frac{1}{2}A < BC$. If the inequality sign is replaced by the equality sign there is a single construction of an isoscoles triangle.

Also solved by Eugene Alliot, Brother Aurelius, L. M. Bauer, Mannis Charosh, Edward Fleisher, R. Goormaghtigh, H. Grossman, Theodore Lindquist, Mrs. Elizabeth Nixon, A. Pelletier, W. H. Rasche, H. D. Ruderman, William Sell, Wallace Smith, F. Underwood, Hymen Weisberg, J. M. West, and Roscoe Woods.

3521 [1932, 45]. *Proposed by J. M. West, Pennsylvania State College.*

Show that the equations of the tangents of the circle, $x^2 + y^2 = r^2$, through the external point (a, b) are

$$(ar \pm bl)x + (br \mp at)y = r(a^2 + b^2),$$

where t is the length of the tangents from the point (a, b) to the circle.

Solution by Dorothy McCoy, Belhaven College

The tangent to the circle $x^2+y^2=r^2$ at the point (α, β) on the circle is

$$\alpha x + \beta y = r^2. \tag{1}$$

The tangent at (α, β) will pass through (a, b) if

$$a\alpha + b\beta = r^2 \tag{2}$$

but (α, β) is on the circle; hence

$$\alpha^2 + \beta^2 = r^2. \tag{3}$$

The tangent is perpendicular to the radius at the point of tangency; hence

$$a^2 + b^2 - r^2 = t^2, \tag{4}$$

where t is the length of the tangent.

Solving (2) and (3) for α we get

$$\alpha = \frac{r}{a^2 + b^2}\left\{ar \pm [r^2a^2 - (a^2 + b^2)(r^2 - b^2)]^{1/2}\right\}$$

which becomes by use of (4)

$$\alpha = \frac{r}{a^2 + b^2}(ar \pm bt). \tag{5}$$

Substituting (5) in (2), we obtain

$$\beta = \frac{r}{a^2 + b^2}(br \mp at).$$

Substituting these values of α and β in (1) we have the required equations of the tangents.

Also solved by E. F. Allen, H. T. R. Aude, George A. Baker, C. A. Barnhart, L. M. Bauer, S. F. Bibb, J. H. Butchart, W. B. Campbell, C. S. Carleton, A. G. Clark, J. M. Feld, S. E. Field, Edward Fleisher, J. W. Foust, H. G. Funkhouser, E. A. Goodhue, R. Goormaghtigh, D. F. Gunder, J. D. Hill, C. R. Hillard, H. S. Kaltenborn, J. J. Knox, Charlotte M. Lacy, C. H. Lehmann, J. D. Leith, Theodore Lindquist, Roy MacKay, J. B. Meyer, W. K. Morrill, W. V. Parker, A. Pelletier, A. R. Randall, A. W. Rankin, C. C. Richtmeyer, H. A. Robinson, H. D. Ruderman, Rafael Sanchez-Diaz, Hazel E. Schoonmaker, H. L. Schug, Wm. Sell, Mrs. Ruth B. Smyth, R. C. Staley, F. Underwood, Hymen Weisberg, Paul Wernicke, F. G. Williams, F. L. Wilmer, Roscoe Woods, and R. C. Yates.

3523 [1932, 45]. *Proposed by Dewey C. Duncan, University of California.*

Solve completely the equation

$$x^7 + Ax^6 + Bx^4 + Cx^2 + D = 0,$$

A, B, C, D, being unknown constants and given that the equation has a double root and one triple root.

Solution by F. Underwood, University College, Nottingham, England

Denote the roots of the given equation by

$$\alpha, \alpha, \alpha, \beta, \beta, \gamma, \delta.$$

CASE 1. When $D=0, \beta=0$, and the equation

$$x^5 + Ax^4 + Bx^3 + C = 0 \tag{2}$$

has roots α, α, α, γ, δ. Putting $\sum\alpha = -A$; $\sum\alpha\gamma = 0$; $\sum\alpha\gamma\delta = -B$; $\sum\alpha^2\gamma\delta = 0$; $\alpha^3\gamma\delta = -C$, we find that $\gamma+\delta = -9\alpha/8$, $\gamma\delta = 3\alpha^2/8$, $A = -15\alpha/8$, $B = 5\alpha^3/4$, $C = -3\alpha^5/8$, and hence the equation,

$$8x^7 - 15\alpha x^6 + 10\alpha^3 x^4 - 3\alpha^5 x^2 = 0,$$

has roots α, α, α, 0, 0, $\alpha\{-9 \pm (-15)^{1/2}\}/16$ for all values of α.

Case 2. When $D \neq 0$, the equation has no zero roots. Proceeding as before, but using only

$$\sum\alpha\beta = \sum\alpha\beta\gamma\delta = \sum\alpha^2\beta^2\gamma\delta = 0,$$

(i.e. the three relations which do not involve the constants A, B, C, D), and eliminating γ, δ we find that a necessary condition is

$$3\alpha^5 + 18\alpha^4\beta + 43\alpha^3\beta^2 + 48\alpha^2\beta^3 + 24\alpha\beta^4 + 4\beta^5 = 0,$$

which reduces to

$$(\alpha + \beta)^2(3\alpha^3 + 12\alpha^2\beta + 16\alpha\beta^2 + 4\beta^3) = 0.$$

It is found immediately that an equation of the type required cannot exist for $\alpha+\beta=0$, and so, putting $\alpha/\beta = z$, we have

$$3z^3 + 12z^2 + 16z + 4 = 0. \tag{3}$$

Substituting $z = y/3$, followed by $y = t-4$, this equation is reduced to $t^3 - 28 = 0$, so that the only real root of equation (3) is $\{(28)^{1/3} - 4\}/3$, giving the only possible real value of α/β. If we regard this as giving α in terms of β, γ and δ are expressed in terms of β by means of the three relations previously used, and the other four relations then give A, B, C, D in terms of β, so that an equation of the required form is obtained in the general case.

It may be remarked that the actual evaluation of these constants necessarily involves long and clumsy work.

Note by the Editors. In regard to the last remark by the solver, it may be

added that the results do not appear to be of sufficient interest or importance to justify the labor of obtaining them.

3524 [1932, 46]. *Proposed by W. H. Echols, University of Virginia.*

At the corners of any equilateral triangle ABC let there be hinged three equilateral triangles ALM, BNO, CPQ of any sizes or positions.

Then will the midpoints of each of the sets of the three segments

$$(LQ, OP, MN), \quad (BQ, CN, OP),$$
$$(LB, OA, MN), \quad (AP, CM, LQ)$$

be the corners of an equilateral triangle.

This problem has been covered as a special case in a paper by J. R. Musselman (This MONTHLY, vol. 39, 1932, p. 290) and in the note by R. Goormaghtigh in this issue, p. 535. Also solved by E. Alliott, J. H. Butchart, W. Sell, Paul Wernicke, and F. L. Wilmer.

3526 [1932, 46]. *Proposed by R. E. Gaines, University of Richmond.*

If a triangle PQR be inscribed in the ellipse $x^2/a^2+y^2/b^2=1$, and if PQ and PR are tangent to the hyperbola, $x^2/a^2-y^2/b^2=1$, the locus of the pole of QR with respect to the ellipse is identical with the locus of the pole of QR with respect to the hyperbola.

Solution by Roscoe Woods, The State University of Iowa

Let the coordinates of P be (x_1, y_1). Suppose the equations of the lines PQ and PR to be $y-y_1=m_i(x-x_1)$ $(i=1, 2)$. Set $X=x-x_1$ and $Y=y-y_1$ and multiply these two equations together. We have

$$Y^2 - (m_1 + m_2)XY + m_1m_2X^2 = 0. \tag{1}$$

The line $y-y_1=m(x-x_1)$ will be tangent to the hyperbola $x^2/a^2-y^2/b^2=1$ when $y_1-mx_1=\pm(a^2m^2-b^2)^{1/2}$ or when $m^2(x_1^2-a^2)-2mx_1y_1+b^2+y_1^2=0$. If m_1 and m_2 are the roots of the equation just written, we may write equation (1) in the form

$$X^2(y_1^2 + b^2) - 2x_1y_1XY + Y^2(x_1^2 - a^2) = 0. \tag{2}$$

This is the equation of the lines PQ and PR which pass through the new origin $X=0$, $Y=0$.

Replacing x and y by $X+x_1$ and $Y+y_1$, respectively in the equation of the ellipse we have

$$b^2X^2 + a^2Y^2 + 2b^2x_1X + 2a^2y_1Y = 0. \tag{3}$$

Let the equation of QR be $AX+BY=C$. If we make equation (3) homogeneous by means of this equation, we have

$$X^2(b^2C + 2b^2x_1A) + Y^2(a^2C + 2a^2y_1B) + 2(b^2x_1B + a^2y_1A)XY = 0. \tag{4}$$

This equation is the equation of the pair of lines joining the origin ($X=0$, $Y=0$) and the intersections of the line QR and the ellipse. Hence the equations (2) and (4) represent the same lines and must be identical. Equating coefficients of like powers we have after a short calculation $A:B:C=x_1/2a^2:-3y_1/2b^2:2y_1^2/b^2$. Hence the equation of QR becomes, after replacing X and Y by their values $x-x_1$ and $y-y_1$ respectively,

$$xx_1/a^2 - 3yy_1/b^2 = 1. \tag{5}$$

The poles of the line (5) with respect to the ellipse and the hyperbola are $(x_1, -3y_1)$ and $(x_1, 3y_1)$ respectively. Since the point (x_1, y_1) is any point on the ellipse, the loci of these poles are identical. It is easily found to be $x^2/a^2+y^2/9b^2=1$. This proves the proposition.

Also solved by J. H. Butchart, W. B. Campbell, Rufus Crane, F. Underwood, and Paul Wernicke.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this Department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The quadrennial International Congress of Mathematicians was held as Zurich, Switzerland, September 4–12, 1932. Twenty general lectures in variout fields of mathematics were given by invited representatives of ten different nationalities. The United States was represented in able addresses given by Professors Marston Morse of Harvard University and J. W. Alexander of Princeton University. Four half days were devoted to sectional meetings at which results of recent research were given in about 240 briefer papers. The full text of the invited lectures and extended abstracts of the contributed papers will appear in the printed volume of the Congress. The Congress was characterized by reports of extensive current research and by profitable personal contacts between mathematicians of different countries, rather than by announcements of any epoch-making discoveries.

The International Commission on the Teaching of Mathematics held important sessions under the presidency of Professor David Eugene Smith. Doctor Smith mapped out a program for the next few years. He urged the great need for the careful collection of information as to recent developments in mathematical teaching the world over, in continuation of the well-known series of books issued through the activity of the Commission beginning about 1907; he instanced in particular our ignorance of what is being done in connection with the tremendous experiment of the Russian people. Professor Jacques Hadamard is to serve as president of the Commission for the next four years.

A session was held of the International Mathematical Union. Because of the unsatisfactory character of its work and because of its decidedly political nature, representing as it does the governments rather than the scientific or-

ganizations, the Union has been subjected to severe criticism; and at the insistence of American, English and Danish mathematicians it was agreed to disband the Union and to ask the Congress to appoint a committee to investigate the need for a continuing body of the sort.

Approximately 650 mathematicians with 200 persons accompanying them came to the Congress from 41 countries. Of these there were 68 mathematicians from the United States and Canada, with 30 accompanying them. Elaborate provisions were made for the social entertainment of the visitors, including several half-days given to outings in that general region of Switzerland, a festal evening of welcoming speeches and spectacular dancing at the Stadttheater, and a low-cost excursion to the Jungfraujoch following the week of the Congress.

The Congress accepted the invitation to hold the next Congress at Oslo, Norway, four years hence.

A full account of the Congress appears in the Bulletin of the American Mathematical Society.

The first number of a new quarterly journal of mathematics, Scripta Mathematica, has just appeared. It is published by Yeshiva College at Amsterdam Avenue and 186th Street, New York. Jekuthiel Ginsburg is the editor. Raymond Clare Archibald, Cassius Jackson Keyser, Louis Karpinski, Gino Loria, Lao Genevra Simons and David Eugene Smith are the other members of the editorial staff. The journal is to be devoted to the interests of the philosophy, history, and expository treatment of mathematics.

Each number will contain approximately ninety-six pages of reading matter described by the editor as follows:

"The pages of the periodical will be devoted chiefly to the history and philosophy of mathematics. The expositorial treatment of mathematics will be included for the purpose of giving to the reader a knowledge of what is being done at present in various branches of the subject and of the history of mathematics in the making.—A special effort will be made to have the articles free from such technicalities as would repel the intelligent reader who has not had a thorough training in mathematics."

The American Institute of Physics announces that it will shortly begin publication of a new periodical, the Journal of Chemical Physics, under the editorship of Professor H. C. Urey, of Columbia University.

The American Mathematical Society appointed the following as its official delegates to the International Mathematical Congress at Zurich: Edward Kasner, C. N. Moore, R. G. D. Richardson, E. B. Stouffer, and J. D. Tamarkin.

Dr. Joseph Slepian, of the Westinghouse Electric and Manufacturing Company, has received a John Scott award, for his work in connection with gases and fundamental inventions involving these discoveries.

Professor R. A. Millikan has received an honorary doctorate of science from Harvard University.

Professor J. M. Thomas, of Duke University, has been elected chairman of the mathematics section of the North Carolina Academy of Sciences, and Professor Helen Barton, of North Carolina College, secretary of that section.

Professor Oswald Veblen, of Princeton University, represented the American Association for the Advancement of Science at the meeting of the British Association at York, August 31 to September 7, 1932.

Dr. O. J. Farrell has been appointed assistant professor of mathematics at Union College.

Assistant Professor J. H. Fithian has been promoted to an associate professorship of mathematics at the Newark College of Engineering.

Dr. H. H. Germond has been appointed assistant professor of mathematics at the University of Florida.

Assistant Professor R. F. Graesser, of the University of Arizona, has been promoted to an associate professorship of mathematics.

R. A. Hefner has been promoted to an assistant professorship at the Georgia School of Technology.

Assistant Professor L. D. Hemenway has been promoted to an associate professorship at Simmons College.

Dr. G. B. Price has been appointed instructor in mathematics at Union College.

Mr. A. E. Whitford, of the University of Wisconsin, has been appointed professor of mathematics at Alfred University.

Dr. Kamcheung Woo has been appointed professor of mathematics at Sun Yatsen University, Canton, China.

Professor W. M. Brodie, of the Virginia Polytechnic Institute, died in April, 1932 of pneumonia. He had been a member of the Association since 1920.

Professor C. N. Dickinson, of Hollins College, a charter member of the Association, died May 25, 1932.

Associate Professor James E. Donahue, of the University of Vermont, died suddenly of cerebral hemorrhage on August 13, 1932, at the age of fifty-two years. He was a charter member of the Association.

Dr. Harold Jacoby, retired professor of astronomy at Columbia University, died July 20, 1932, at the age of sixty-seven. Professor Jacoby was a member of the American Mathematical Society at its organization as the New York Mathematical Society, and was its first treasurer.

Alfred H. Jekel, president of the Colorado Clay and Mining Co., Boulder, a member of the Association since 1925, was killed in an automobile accident near Pine, Colo., March 8, 1932.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Seventeenth Annual Meeting of the Association, Atlantic City, N.J., Dec. 27-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

ILLINOIS, Urbana, May 6-7.
INDIANA, Indianapolis, May 6-7.
IOWA, Cedar Falls, April 29-30.
KANSAS, Topeka, Feb. 13.
KENTUCKY, Lexington, May.
LOUISIANA-MISSISSIPPI, Oxford, Miss., March 11-12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., Dec. 3.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA, River Falls, Wis., May 7.
MISSOURI.
NEBRASKA, Omaha, May 6-7.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo., April 15-16.
SOUTHEASTERN, Gainesville, Fla., Mar. 18-19.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE TWENTY-FIRST MEETING OF THE IOWA SECTION

The twenty-first meeting of the Iowa Section of the Mathematical Association of America was held with the Iowa Academy of Science at Iowa State Teachers College, Cedar Falls, Iowa, on April 29 and 30, 1932. The meetings were held in Room 214, Administration Building.

The attendance was about thirty-five, including the following twenty-one members of the Association: F. A. Brandner, L. M. Coffin, I. S. Condit, N. B. Conkwright, A. T. Craig, C. W. Emmons, Dora E. Kearney, F. M. McGaw, J. V. McKelvey, J. F. Reilly, H. L. Rietz, B. D. Roberts, Fred Robertson, W. J. Rusk, E. R. Smith, G. W. Snedecor, C. W. Strom, J. S. Turner, L. E. Ward, C. W. Wester, Roscoe Woods.

The Section chairman, Professor C. W. Strom, presided at both the Friday afternoon and Saturday morning sessions, relieved for a time by the vice-chairman, Professor B. D. Roberts. Dinner was enjoyed together Friday evening in the Crystal Room of Bartlett Hall, at the close of which the chairman of the local committee, Professor Condit, and the newly elected officers made brief remarks. The officers elected for 1932–1933 are as follows: Chairman, L. M. Coffin, Coe College; Vice-Chairman, Julia T. Colpitts, Iowa State College; Secretary-Treasurer, J. F. Reilly, University of Iowa.

The program consisted of fifteen papers, as follows:

1. "A generalization of the probability integral" by Professor E. R. Smith, Iowa State College.

2. "Discussion of the value of daily written work in mathematics" by Professor Roscoe Woods, University of Iowa.

3. "The utility of analysis of variance in biological research" by Professor G. W. Snedecor, Iowa State College.

4. "Analysis of variance in a $2 \times n$ table with disproportionate frequencies" by Professor A. E. Brandt, Iowa State College, by invitation.

5. "A sampling test of the technique of analyzing variance in a $2 \times n$ table with disproportionate frequencies" by Bernice Brown, Iowa State College, by invitation.

6. "Some conics with names" by Professor Roscoe Woods, University of Iowa.

7. "Complete systems under certain finite groups" (second paper) by Professor C. W. Strom, Luther College.

8. "The function idea for undergraduates" by Professor C. W. Wester, Iowa State Teachers College.

9. "A set of determinants analogous to those of Tchebychef for the factorization of large numbers" by Professor J. S. Turner, Iowa State College.

10. "A relation between graduate and undergraduate work" by Professor B. D. Roberts, Parsons College.